APPLICATIONS OF DUALITY THEORY TO COUSIN COMPLEXES

SURESH NAYAK AND PRAMATHANATH SASTRY

Abstract. We use the anti-equivalence between Cohen-Macaulay complexes and coherent sheaves on formal schemes to shed light on some older results and prove new results. We bring out the relations between a coherent sheaf $\mathscr{M}$ satisfying an S_2 condition and the lowest cohomology $\mathscr{N}$ of its "dual" complex. We show that if a scheme has a Gorenstein complex satisfying certain coherence conditions, then in a finite étale neighborhood of each point, it has a dualizing complex. If the scheme already has a dualizing complex, then we show that the Gorenstein complex must be a tensor product of a dualizing complex and a vector bundle of finite rank. We relate the various results in [S] on Cousin complexes to dual results on coherent sheaves on formal schemes.

1. Introduction

The theme which lurks behind the various results in this paper is the (anti) equivalence between Cohen-Macaulay complexes and coherent sheaves proven in [LNS, p. 108, Prop. 9.3.1 and Cor. 9.3.2] and restated here in Proposition 2.5.1 and Proposition 2.7.4. The Cohen-Macaulay complexes we just referred to are with respect to a fixed codimension function and satisfy certain coherence conditions, which for ordinary schemes amount to requiring that all cohomology sheaves are coherent. In all the sections, except Section 6, the formal schemes involved are also required to satisfy conditions—e.g. they should carry "c-dualizing complexes" (see Definition 2.3.1 below).

This anti-equivalence is the unifying thread that runs through the three main topics of this paper. It was first observed by Yekutieli and Zhang in [YZ, Thm. 8.9] for ordinary schemes of finite type over a regular scheme, and later in greater generality by Lipman and the authors in [LNS]. We first give a short description of each topic we deal with before embarking on a more detailed discussion putting our results in context. Here is the brief version:

1) We explore symmetries between a coherent sheaf (on an ordinary scheme) satisfying an "S_2 condition" with respect to a codimension function (cf. Definition 3.2.1) and an associated "dual" coherent sheaf (which also is shown to satisfy the same S_2 condition). The example to keep in mind is the symmetry between the structure sheaf of an S_2 scheme and a canonical module on the scheme (cf. [DT, p. 19, Thm. 1.4] and [Kw, Thm. 4.4]).

2) We give a relationship between Gorenstein complexes and dualizing complexes (both with respect to a fixed codimension function).

3) We find an alternate approach to some of the results in [S] when our Cousin complexes involved satisfy certain coherence conditions (which, as before, translate on an ordinary scheme to usual coherence conditions). And in this approach we do

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not need to assume that the maps involved (between formal schemes) are composites of compactifiable maps. It was A. Yekutieli who made the suggestion (to the second author) that the results in [S] should be re-examined in light of the above mentioned duality between Cohen-Macaulay complexes and coherent sheaves. It should, however, be pointed out that the proofs in the present paper only give an illusion of being simpler for we need the deeper results of [S], which deals with a larger category. However, our proofs are illuminating, since they interpret operations on Cousin complexes in terms of natural operations on the dual category of coherent sheaves.

Let us examine each of these topics in somewhat greater detail. All schemes involved (formal or ordinary) are assumed to be noetherian and carrying a c-dualizing complex (forcing them to be of finite Krull dimension). We use the following notations

- For a scheme $\mathscr{X}$, and $\mathscr{O}_{\mathscr{X}}$-modules $\mathscr{A}$ and $\mathscr{B}$, we often write $\mathrm{Hom}(\mathscr{A}, \mathscr{B})$ and $\mathscr{H}om(\mathscr{A}, \mathscr{B})$ for $\mathrm{Hom}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{A}, \mathscr{B})$ and $\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(\mathscr{A}, \mathscr{B})$ respectively (here $\mathscr{H}om$ is the sheafified version of Hom).
- Similarly, when no confusion is likely to arise, we write $\mathscr{A}\otimes\mathscr{B}$ for $\mathscr{A}\otimes_{\mathscr{O}_{\mathscr{X}}}\mathscr{B}$.
- For an $\mathscr{O}_{\mathscr{X}}$ complex $\mathscr{F}$, and an integer p, $\mathscr{F}^p$ will denote the p-th graded piece of $\mathscr{F}$.
- For two $\mathscr{O}_{\mathscr{X}}$ complexes $\mathscr{F}$ and $\mathscr{G}$, $\mathrm{Hom}^{\bullet}(\mathscr{F}, \mathscr{G})$ denotes $\mathrm{Hom}^{\bullet}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{G})$, i.e., $\mathrm{Hom}^{\bullet}$ gives a complex of $\Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$-modules. In contrast, $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ denotes the "external Hom", i.e. $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_{\mathbf{C}(\mathscr{X})}(\mathscr{F}, \mathscr{G})$ the $\Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$-module of $\mathscr{O}_{\mathscr{X}}$ co-chain maps from $\mathscr{F}$ to $\mathscr{G}$. It is well known that the latter is the module of 0–cocycles of the former.
- With $\mathscr{F}$ and $\mathscr{G}$ as above, $\mathscr{H}om^{\bullet}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{G})$ and $\mathscr{H}om_{\mathbf{C}(\mathscr{X})}(\mathscr{F}, \mathscr{G})$ denote the sheafified versions of $\mathrm{Hom}^{\bullet}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{G})$ and $\mathrm{Hom}_{\mathbf{C}(\mathscr{X})}(\mathscr{F}, \mathscr{G})$ respectively. As before, when the context is clear, we write $\mathscr{H}om^{\bullet}$ for $\mathscr{H}om^{\bullet}_{\mathscr{O}_{\mathscr{X}}}$ and $\mathscr{H}om$ for $\mathscr{H}om_{\mathbf{C}(\mathscr{X})}$. The relationship between $\mathscr{H}om^{\bullet}$ and $\mathscr{H}om$ is analogous to the relation between $\mathrm{Hom}^{\bullet}$ and Hom. We identify the $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ with the sheaf of 0-cocycles in $\mathscr{H}om^{\bullet}(\mathscr{F}, \mathscr{G})$.

1.1. **Δ-S_2 complexes.** Let X be an ordinary scheme and let $\mathscr{R}$ be a dualizing complex on X which we assume (without loss of generality) is residual. Let $\Delta\colon |X| \to \mathbb{Z}$ be the associated codimension function (so that $\mathscr{R} = E_{\Delta}\mathscr{R}$, where E_{Δ} is the Cousin functor associated with Δ (see §§ 2.3 below)). Recall that X is S_2 if and only if the natural map $\mathscr{O}_X \to E_{\mathscr{H}}(\mathscr{O}_X)$ gives an isomorphism on applying H^0, where, $E_{\mathscr{H}}(\mathscr{O}_X)$ is the Cousin complex of $\mathscr{O}_X$ with respect to the "height filtration" $\mathscr{H} = (H_i)$ given by $H_i = \{x \in X \mid \dim \mathscr{O}_{X,x} \geq i\}$. One defines the notion of a Δ-S_2 module along the above lines (cf. Definition 3.2.1). Let $\mathscr{M}$ be such a module, which by definition is coherent. Let $\mathscr{N} = \mathscr{H}om(E_{\Delta}\mathscr{M}, \mathscr{R})$. We show that $\mathscr{N}$ is also coherent and $\mathscr{M}$ and $\mathscr{N}$ share the following symmetries, where "=" denotes functorial isomorphisms (cf. Theorem 3.2.8).

(i) $\mathscr{M} = \mathscr{H}om(E_{\Delta}\mathscr{N}, \mathscr{R})$. (Note: $\mathscr{N} := \mathscr{H}om(E_{\Delta}\mathscr{M}, \mathscr{R})$.)
(ii) $E_{\Delta}\mathscr{M} = \mathscr{H}om^{\bullet}(\mathscr{N}, \mathscr{R})$, $E_{\Delta}\mathscr{N} = \mathscr{H}om^{\bullet}(\mathscr{M}, \mathscr{R})$.
(iii) $\mathscr{M} = H^0(\mathscr{H}om^{\bullet}(\mathscr{N}, \mathscr{R}))$, $\mathscr{N} = H^0(\mathscr{H}om^{\bullet}(\mathscr{M}, \mathscr{R}))$.

If $S_2(\Delta)$ is the category of all Δ-S_2 modules (viewed as a full subcategory of the category of coherent $\mathscr{O}_X$-modules) and coz^2_{Δ} represents the essential image of $S_2(\Delta)$ under E_{Δ}, then the situation is summarized by the following weakly "commutative"

diagram.

$$\begin{tikzcd} S_2(\Delta) \arrow[r, leftrightarrow, "\text{dualize}"] \arrow[d, leftrightarrow, "H^0\uparrow" left, "\downarrow E_\Delta" right] & \mathrm{coz}^2_\Delta \arrow[d, leftrightarrow, "E_\Delta\uparrow" left, "\downarrow H^0" right] \\ \mathrm{coz}^2_\Delta \arrow[r, leftrightarrow, "\text{dualize}" below] & S_2(\Delta) \end{tikzcd} \tag{1.1.1}$$

In terms of the discussion above the diagram, if $\mathscr{M} \in S_2(\Delta)$ is an object in the northwest vertex, then its "dual", $\mathscr{N} \in S_2(\Delta)$ occurs in the southeast vertex. If $\Delta(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$ ($\mathfrak{p} \in \mathrm{Spec}(A)$), and $\mathscr{M} = \mathscr{O}_X$ is Δ-S_2, then $\mathscr{N}$ is a canonical module and the above relations have been established by Dibaei, Tousi [DT] and Kawasaki [Kw] as we pointed out earlier.

1.2. **Gorenstein complexes.** The study of Gorenstein modules over a local ring A was initiated by Sharp in [Sh1] where their first properties were established. A non-zero finitely generated A-module G is Gorenstein if—when regarded as a complex—it is a Gorenstein complex in the sense of [Hrt, p. 248] (see (a), (b), (c) below for an extension to formal schemes). In commutative algebraic terms, a non-zero finitely generated A-module is Gorenstein if its Cousin complex (with respect to the height filtration) is an injective resolution of G. If A admits a Gorenstein module, Sharp shows, A is Cohen-Macaulay, the associated height function is a codimension function on $X = \mathrm{Spec}(A)$, $\mathrm{Hom}(G, G)$ is free of rank r^2, $r > 0$. The positive integer r is called the Gorenstein rank of G. The module G (regarded as a complex) is a dualizing complex if and only if $r = 1$. If A has a Gorenstein module then it has one of rank $r = 1$ if and only if A is the homomorphic image of a Gorenstein ring, if and only if A has a dualizing complex. In [FFGR], Fossum, Foxby, Griffith and Reiten show that if G is Gorenstein of minimal rank, then every Gorenstein module on A is of the form G^s for some $s \geq 1$. This last result was anticipated in [Sh2] by Sharp in the instance when A is a complete Cohen-Macaulay ring, so that, by Cohen's structure theorem, A is the homomorphic image of a Gorenstein ring, and whence has a Gorenstein module of rank $r = 1$, necessarily of minimal rank. (Cf. also [Sh4] for related results.) In addition to the above mentioned results in [FFGR], Fossum *et. al.* also show that if A has a Gorenstein module, then some standard étale neighborhood of A has a Gorenstein module of rank $r = 1$ (i.e. a Gorenstein module which is also a dualizing complex).

Consider a pair $(\mathscr{X}, \Delta)$ where $\mathscr{X}$ is a formal scheme, universally catenary, of finite Krull dimension and Δ a codimension function on $\mathscr{X}$. Let $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$ be the full subcategory of $\mathbf{D}(\mathscr{X})$ (the derived category of the category of complexes of $\mathscr{O}_{\mathscr{X}}$-modules) consisting of objects whose cohomology sheaves are coherent, and let $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ denote its essential image under the functor $\mathbf{R}\varGamma'_{\mathscr{X}}|_{\mathbf{D}_{\mathrm{c}}}$, where (with $\mathscr{I}$ a defining ideal of $\mathscr{X}$)

$$\varGamma'_{\mathscr{X}} := \varinjlim_{n} \mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(\mathscr{O}_{\mathscr{X}}/\mathscr{I}^n, -).$$

(If $\mathscr{X}$ is ordinary, then $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X}) = \mathbf{D}_{\mathrm{c}}(\mathscr{X})$.) A complex $\mathscr{G}$ is said to be c-Gorenstein on $(\mathscr{X}, \Delta)$ (or Δ-Gorenstein) if

(a) $\mathscr{G} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$.
(b) $\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G} \cong E_\Delta \mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}$ in $\mathbf{D}(\mathscr{X})$, i.e. $\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}$ is Cohen-Macaulay on $(\mathscr{X}, \Delta)$.
(c) $E_\Delta \mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}$ consists of $\mathcal{A}_{\mathrm{qct}}(\mathscr{X})$-injectives, where $\mathcal{A}_{\mathrm{qct}}(\mathscr{X})$ is as in §§ 2.2.

For the rest of this discussion, for simplicity, we assume that our complex $\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}$ is non-exact on every connected component of $\mathscr{X}$, equivalently $E\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{G} \neq 0$ on any connected component of $\mathscr{X}$. In this paper, using this result, we show that if $\mathscr{X}$ has a c-dualizing complex, then

$$(*) \qquad \mathscr{G} \cong \mathscr{D} \otimes \mathcal{V}$$

where $\mathscr{D}$ is a c-dualizing complex whose associated codimension function is Δ and $\mathcal{V}$ is a coherent locally free $\mathscr{O}_{\mathscr{X}}$-module (cf. Theorem 6.2.6). Note that it follows that $\mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{G},\mathscr{G})$ is isomorphic in $\mathbf{D}(\mathscr{X})$ to $\mathcal{V}^* \otimes \mathcal{V}$, i.e. to a coherent locally free $\mathscr{O}_{\mathscr{X}}$-module of rank r^2, where r is the rank of $\mathcal{V}$. Since $\mathscr{X}$ is not assumed to be connected, we have to interpret r as a locally constant, positive integer valued function.

Suppose we drop the assumption that $\mathscr{X}$ has a c-dualizing complex. Can r (the "rank" of $\mathscr{G}$) still be defined? In Proposition 6.2.5 (and its proof) we show that $\mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{G},\mathscr{G})$ is isomorphic (in $\mathbf{D}(\mathscr{X})$) to a coherent locally free sheaf $\mathcal{W}$ of rank r^2 where r is a positive integer valued function. In fact, for a point $x \in \mathscr{X}$, $r(x)$ is the number of copies of the injective hull $E(x)$ of the residue field $k(x)$ (thought of as a $\mathscr{O}_{\mathscr{X},x}$-module) in the injective module $G(x) = \mathrm{H}_x^{\Delta(x)}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G})$. The result implies that this number (of copies of $E(x)$ in $G(x)$) is constant on connected components of $\mathscr{X}$, something which is not *a priori* obvious. Further, when $r = 1, \mathscr{G}$ is c-dualizing. We study the (possibly) non-commutative $\mathscr{O}_{\mathscr{X}}$-algebra $\mathscr{A} = \mathscr{H}om(E\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}, E\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G})$ (isomorphic as a coherent sheaf to $\mathcal{W}$), for it sheds light on the existence of étale open sets of $\mathscr{X}$ on which $\mathscr{G}$ "untwists" and reveals itself in the form (*). In fact one can show that $\mathscr{A}$ is a sheaf of Azumaya algebras (see Proposition 6.3.8), whose splitting implies the existence of a dualizing complex (see Theorem 6.3.9). This generalizes [FFGR, p. 209, Cor. (4.8)]. One consequence is this: if $(\mathscr{X}, \Delta)$ has a c-Gorenstein complex, then $E_{\Delta}(\mathscr{F}) \in \mathbf{D}_{\mathrm{c}}^*$ if $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}^*$ (cf. Theorem 6.3.10 (b)). In particular if $\mathscr{X}$ is an ordinary scheme (so that $\mathbf{D}_{\mathrm{c}}^* = \mathbf{D}_{\mathrm{c}}$) possessing a c-Gorenstein complex and if $\mathscr{F}$ has coherent cohomology, then so does its Cousin complex with respect Δ.

Now suppose $\mathscr{X} = \operatorname{Spec}(A)$, where A is a local ring of dimension d. Suppose further that $\mathscr{X}$ has a dualizing complex and that $\mathscr{O}_{\mathscr{X}}$ is S_2. It is not hard to show that this forces $\mathscr{O}_{\mathscr{X}}$ to be Δ-S_2 where Δ is the codimension function $\mathfrak{p} \mapsto d - \dim A/\mathfrak{p}$. Moreover, in this case it is well-known (see, for example, [DT, p. 23, Rmk.,2.1]) that Δ is the "height function" $\mathfrak{p} \mapsto \mathrm{ht}_A(\mathfrak{p})$. Let $\mathscr{D}$ be a dualizing complex whose associated codimension function is Δ (under our hypothesis, such a $\mathscr{D}$ exists). Let $\mathscr{K} := H^0(\mathscr{D})$. Now (*) combined with Theorem 3.2.8 gives us that if $\mathscr{G}$ is Gorenstein with respect to the Δ, then $\mathscr{N} := H^0(\mathscr{G})$ is also S_2 and $\mathscr{N} = \mathscr{K} \otimes \mathcal{V}$. We believe this gives a more natural interpretation of [Db, p. 125, Thm. 3.3].

In Subsection 6.4 we discuss (very briefly) the relationship between various results in this paper (especially Theorem 6.2.6 and Theorem 6.3.10) and Gorenstein *modules*.

1.3. **Duality theory.** The paper [S] is concerned with studying "the gap" between the Cousin complex $f^{\sharp}\mathscr{F}$ constructed in [LNS] and the twisted inverse image $f^!\mathscr{F}$ (for a map $f\colon \mathscr{X} \to \mathscr{Y}$ which is a composite of compactifiable maps and a torsion Cousin complex $\mathscr{F}$ on $\mathscr{Y}$). This is done via a functorial map $\gamma^!_f(\mathscr{F})\colon f^{\sharp}\mathscr{F} \to f^!\mathscr{F}$ which is defined for a pseudo-proper map by the universal properties of $f^!$ (since Sastry shows that for such a map there is a map of complexes $\mathrm{Tr}_f\colon f_*f^{\sharp}(\mathscr{F}) \to \mathscr{F}$),

and then for maps f which are composites of compactifiable maps. The main result is that $E(\gamma^!_f)$ is an isomorphism, and the hardest technical step is in showing that $\gamma^!_f$ is an isomorphism when f is smooth. From this a number of results follow: among them the just mentioned fact that $E(\gamma^!_f)$ is an isomorphism, $\gamma^!_f$ is a functorial isomorphism when f is flat, and that $\gamma^!_f(\mathscr{F})$ is an isomorphism when $\mathscr{F}$ is residual (or more generally, in the language of the present paper, when $\mathscr{F}$ is t-Gorenstein). This last result is crucial in establishing an explicit isomorphism between the functor $f^!$ (restricted to complexes satisfying certain coherence conditions) and a functor $f^{(!)}$ defined along the lines of Grothendieck's original treatment of his duality on ordinary schemes, as laid out in [Hrt] (i.e., by first "dualizing" on the base using a residual complex $\mathscr{R}$, then applying $\mathbf{L}f^*$, and then "dualizing" on the source using $f^\sharp(\mathscr{R})$, with the identification of $f^\sharp(\mathscr{R})$ with $f^!\mathscr{R}$ via $\gamma_f(\mathscr{R})$ being needed crucially). Now, Cousin complexes are equivalent to Cohen-Macaulay complexes. Therefore there is a duality (i.e. an anti-equivalence) between Cousin complexes in $\mathbf{D}^*_{\mathrm{c}}$ and coherent sheaves. It is natural to ask for dual notions corresponding to $f^\sharp$ and $f^!$ (restricted to Cousins in $\mathbf{D}^*_{\mathrm{c}}$). We show that the corresponding functors on coherent sheaves are f^* and $\mathbf{L}f^*$. Theorem 4.4.3 (iii) and (iv) together with Theorem 5.3.3 should be regarded as the precise formulation of this statement. Thus, if $\mathscr{F}$ is Cousin on $\mathscr{Y}$ and in $\mathbf{D}^*_{\mathrm{c}}(\mathscr{Y})$, and $\mathscr{M}$ the associated coherent sheaf (under our duality), then the gap between $f^\sharp\mathscr{F}$ and $f^!\mathscr{F}$ is equivalent—in the dual situation—to the gap between $f^*\mathscr{M}$ and $\mathbf{L}f^*\mathscr{M}$. The comparison map $\gamma^!_f\colon f^\sharp \to f^!$ corresponds to the natural transformation $\mathbf{L}f^* \to f^*$ on coherent $\mathscr{O}_{\mathscr{Y}}$-modules. If f is flat, this means that the gap can be closed for all Cousins $\mathscr{F}$ in $\mathbf{D}^*_{\mathrm{c}}(\mathscr{Y})$ (i.e. for all coherent $\mathscr{M}$ on $\mathscr{Y}$) and vice-versa. This gives a natural interpretation of the result in [S, p. 182, 7.2.2] (cf. Theorem 4.4.4 together with Theorem 5.3.3). In general, the condition that $f^\sharp\mathscr{F} \cong f^!\mathscr{F}$ imposes conditions on the pair $(f, \mathscr{F})$, whence on $(f, \mathscr{M})$. We interpret this in terms of Tor-independence (cf. Definition 4.4.1 and Lemma 4.4.2). There are drawbacks to the approach taken in this paper. We have to restrict ourselves to complexes satisfying certain coherence conditions (they should be in $\mathbf{D}^*_{\mathrm{c}}$) and to schemes carrying c-dualizing complexes, whereas Sastry works with any torsion Cousin complex, and without the use of c-dualizing complexes. Even with these restrictions, we point out to the readers that any seeming simplicity in the proofs here is nullified by the realization that we do not use $f^\sharp$, $f^!$ and $\gamma^!_f$ for our "simpler" proofs, but use instead the analogous functors (and natural maps) $f^{(\sharp)}$, $f^{(!)}$, and $\gamma^{(!)}_f$. To show (as we do inTheorem 5.3.3 and Diagram (5.3.2.1)) that the analogous functors and maps are actually isomorphic, one requires some of the deeper results in [S]. So the point of the results on Grothendieck duality in this paper is to establish the dictum "$f^\sharp$ is to $f^!$ as f^* is to $\mathbf{L}f^*$" i.e. "$\gamma^!_f$ is dual to $\mathbf{L}f^* \to f^*$". There are gains in doing this, for, after replacing $f^!$ by its variant $f^{(!)}$ [S, § 9], $f^\sharp$ by its variant $f^{(\sharp)}$, and $\gamma^!_f$ by $\gamma^{(!)}_f$, we are able to extend the results in [S] to arbitrary pseudo-finite type maps between the allowed schemes, whereas in [S], Sastry had to restrict himself to maps which were composites of compactifiable maps.

2. Preliminaries

In this paper, all schemes—ordinary or formal—are noetherian.

2.1. **Basic Terminology; Cousin complexes.** A codimension function on a formal scheme $\mathscr{X}$ is a map

$$\Delta\colon |\mathscr{X}| \to \mathbb{Z}$$

such that $\Delta(x') = \Delta(x) + 1$ for every immediate specialization x' of a point $x \in \mathscr{X}$. Here, $|\mathscr{X}|$ denotes the set of points underlying the scheme $\mathscr{X}$. Let $\mathbb{F}$ denote the category whose objects are noetherian universally catenary formal schemes admitting a codimension function and whose morphisms $\mathscr{X}' \to \mathscr{X}$ are formal scheme maps which are *essentially of pseudo-finite type*, i.e., if $\mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$ and $\mathscr{J} \subset \mathscr{O}_{\mathscr{X}'}$ are ideals of definition of $\mathscr{X}$ and $\mathscr{X}'$ respectively, with $\mathscr{J} \supset \mathscr{I}\mathscr{O}_{\mathscr{X}'}$, then the map $\mathbf{Spec}(\mathscr{O}_{\mathscr{X}'}/\mathscr{J}) \to \mathbf{Spec}(\mathscr{O}_{\mathscr{X}}/\mathscr{I})$ is a localization of a finite type map of ordinary schemes.

As in [LNS], we often work with a slightly more refined category $\mathbb{F}_c$ consisting of objects $(\mathscr{X}, \Delta)$, where $\mathscr{X} \in \mathbb{F}$ and Δ is a codimension function on $\mathscr{X}$, whose morphisms $(\mathscr{X}', \Delta') \to (\mathscr{X}, \Delta)$ are maps $f\colon \mathscr{X}' \to \mathscr{X}$ in $\mathbb{F}$ such that for $x' \in \mathscr{X}'$ and $x = f(x')$, $\Delta(x) - \Delta'(x')$ is equal to the transcendence degree of the residue field $k(x')$ of x' over the residue field $k(x)$ of x.

Let $\mathscr{X} \in \mathbb{F}$ and let $x \in \mathscr{X}$. For any abelian group G, i_xG is the extension by zero to $\mathscr{X}$ of the constant sheaf $\overline{G}$ modeled on G, on the closure $\overline{\{x\}}$ of $\{x\}$.

Let $(\mathscr{X}, \Delta) \in \mathbb{F}_c$. A complex $\mathscr{F}$ of $\mathscr{O}_{\mathscr{X}}$-modules is called a *Cousin complex on* $(\mathscr{X}, \Delta)$ or a Δ-Cousin complex (or simply a Cousin complex with respect to Δ) if, for each $n \in \mathbb{Z}$, $\mathscr{F}^n$ is the direct sum of a family of $\mathscr{O}_{\mathscr{X}}$-modules $(i_xF_x)_{x\in\mathscr{X},\Delta(x)=n}$, where F_x is an $\mathscr{O}_{\mathscr{X},x}$-module. We refer the reader to [LNS, pp. 36–44, §§ 3.2 and 3.3] for more elaborate definitions of Cousin (and Cohen-Macaulay) complexes with respect to Δ. We do make fleeting references to Cousin complexes with respect to descending filtration $\mathscr{H} = (H_i)_{i\ge i_\circ}$ of closed sets H_i in $\mathscr{X}$. In this case $\mathscr{F}$ is Cousin with respect to $\mathscr{H}$ if $\mathscr{F}^n$ is the direct sum of a family $(i_xF_x)_{x\in\partial H_i}$, where F_x is an $\mathscr{O}_{\mathscr{X},x}$-module and $\partial H_i = H_i - H_{i+1}$. However our emphasis will always be on the more special Cousin complexes which are associated to a codimension function Δ.

2.2. **Categories of complexes.** For a formal scheme $\mathscr{X}$, let $\mathcal{A}(\mathscr{X})$ be the category of $\mathscr{O}_{\mathscr{X}}$-modules, and $\mathcal{A}_{\mathrm{qc}}(\mathscr{X})$ (resp. $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$, resp. $\mathcal{A}_{\vec{\mathrm{c}}}(\mathscr{X})$) the full subcategory of $\mathcal{A}(\mathscr{X})$ whose objects are the quasi-coherent (resp. coherent, resp. $\varinjlim$'s of coherent) $\mathscr{O}_{\mathscr{X}}$-modules. As in [AJL2, p. 6, 1.2.1], we define the torsion functor $\varGamma'_{\mathscr{X}}\colon \mathcal{A}(\mathscr{X}) \to \mathcal{A}(\mathscr{X})$ by the formula

$$\varGamma'_{\mathscr{X}} := \varinjlim_{n} \mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(\mathscr{O}_{\mathscr{X}}/\mathscr{I}^n,\ -)$$

where $\mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$ is an ideal of definition of $\mathscr{X}$. The definition of $\varGamma'_{\mathscr{X}}$ is independent of the choice of $\mathscr{I}$. Note that $\varGamma'_{\mathscr{X}}$ is a subfunctor of the identity functor.

An object $\mathscr{M} \in \mathcal{A}(\mathscr{X})$ is called a *torsion* $\mathscr{O}_{\mathscr{X}}$-module if $\mathscr{M} = \varGamma'_{\mathscr{X}}(\mathscr{M})$. We denote by $\mathcal{A}_{\mathrm{t}}(\mathscr{X})$ (resp. $\mathcal{A}_{\mathrm{qct}}(\mathscr{X})$) the full subcategory of $\mathcal{A}(\mathscr{X})$ consisting of torsion (resp. quasi-coherent torsion) $\mathscr{O}_{\mathscr{X}}$-modules.

Let $\mathbf{C}(\mathscr{X})$ be the category of $\mathcal{A}(\mathscr{X})$-complexes, $\mathbf{K}(\mathscr{X})$ the associated homotopy category, and $\mathbf{D}(\mathscr{X})$ the corresponding derived category, obtained from $\mathbf{K}(\mathscr{X})$ by inverting quasi-isomorphisms.

For any full subcategory $\mathcal{A}_{\dots}(\mathscr{X})$ of $\mathcal{A}(\mathscr{X})$, denote by $\mathbf{C}_{\dots}(\mathscr{X})$ (resp. $\mathbf{K}_{\dots}(\mathscr{X})$, resp. $\mathbf{D}_{\dots}(\mathscr{X})$) the full subcategory of $\mathbf{C}(\mathscr{X})$ (resp. $\mathbf{K}(\mathscr{X})$, resp. $\mathbf{D}(\mathscr{X})$) whose objects are those complexes whose cohomology sheaves all lie in $\mathcal{A}_{\dots}(\mathscr{X})$, and by $\mathbf{D}^+_{\dots}(\mathscr{X})$ (resp. $\mathbf{D}^-_{\dots}(\mathscr{X})$, resp. $\mathbf{D}^b_{\dots}(\mathscr{X})$) the full subcategory of $\mathbf{D}_{\dots}(\mathscr{X})$ whose

objects are complexes $\mathscr{F} \in \mathbf{D}_{\dots}(\mathscr{X})$ such that the cohomology $H^m(\mathscr{F})$ vanishes for all $m \ll 0$ (resp. $m \gg 0$, resp. $|m| \gg 0$). We often write $\mathbf{D}_{\mathrm{c}}$, $\mathbf{D}_{\mathrm{qct}}$, ... for $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$, $\mathbf{D}_{\mathrm{qct}}(\mathscr{X})$, ... when there is no danger of confusion.

The essential image of $\mathbf{R}\varGamma'_{\mathscr{X}}|_{\mathbf{D}_{\mathrm{c}}}$ is of considerable interest to us, and as in [AJL2, §§ 2.5, p. 24, second paragraph] we denote it by $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$. In greater detail, $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ is the full subcategory of $\mathbf{D}(\mathscr{X})$ such that $\mathscr{E} \in \mathbf{D}^*_{\mathrm{c}}(\mathscr{X}) \Leftrightarrow \mathscr{E} \cong \mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{F}$ with $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$. It is immediate that $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ is a triangulated subcategory of $\mathbf{D}$ or $\mathbf{D}_{\mathrm{qct}}$.

2.3. **Dualizing complexes.** As shown in [AJL2, p. 26, Lemma 2.5.3], the notion of a dualizing complex on an ordinary scheme breaks up into two related notions on a formal scheme. We recall here the definitions and first properties from [AJL2, p. 24, Definition 2.5.1].

Definition 2.3.1. A complex $\mathscr{R}$ is a *c-dualizing complex on* $\mathscr{X}$ if

(1) $\mathscr{R} \in \mathbf{D}^+_{\mathrm{c}}$.
(2) The natural map $\mathscr{O}_{\mathscr{X}} \to \mathbf{R}\mathscr{H}om^\bullet(\mathscr{R}, \mathscr{R})$ is an isomorphism.
(3) There is an integer b such that for every coherent *torsion* sheaf $\mathscr{M}$ and every $i > b$, it holds that $\mathscr{E}xt^i(\mathscr{M}, \mathscr{R}) := H^i\mathbf{R}\mathscr{H}om^\bullet(\mathscr{M}, \mathscr{R}) = 0$.

A complex $\mathscr{R}$ is a *t-dualizing complex* on $\mathscr{X}$ if

(1) $\mathscr{R} \in \mathbf{D}_{\mathrm{t}}{}^+$.
(2) The natural map $\mathscr{O}_{\mathscr{X}} \to \mathbf{R}\mathscr{H}om^\bullet(\mathscr{R}, \mathscr{R})$ is an isomorphism.
(3) There is an integer b such that for every coherent *torsion* sheaf $\mathscr{M}$ and for every $i > b$, $\mathscr{E}xt^i(\mathscr{M}, \mathscr{R}) := H^i\mathbf{R}\mathscr{H}om^\bullet(\mathscr{M}, \mathscr{R}) = 0$.
(4) For some ideal of definition $\mathscr{J}$ of $\mathscr{X}$, $\mathbf{R}\mathscr{H}om^\bullet(\mathscr{O}_{\mathscr{X}}/\mathscr{J}, \mathscr{R}) \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$ (equivalently, $\mathbf{R}\mathscr{H}om^\bullet(\mathscr{M}, \mathscr{R}) \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$ for every coherent torsion sheaf $\mathscr{M}$.)

Yekutieli was the first to consider t-dualizing complexes in [Y], where they were simply called "dualizing". We note from [AJL2, 2.5.3 and 2.5.8] that $\mathscr{X}$ has a c-dualizing complex if and only if $\mathscr{X}$ has a t-dualizing complex which lies in $\mathbf{D}^*_{\mathrm{c}}$. In greater detail, if $\mathscr{R}$ is a c-dualizing complex, then $\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{R} \in \mathbf{D}^*_{\mathrm{c}}$ is a t-dualizing complex. Conversely, if $\mathscr{R}$ is a t-dualizing complex that lies in $\mathbf{D}^*_{\mathrm{c}}$, then $\mathbf{R}\mathscr{H}om^\bullet(\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{O}_{\mathscr{X}}, \mathscr{R})$ is c-dualizing. For an ordinary scheme, $\mathbf{D}_{\mathrm{t}} = \mathbf{D}$ and $\mathbf{D}_{\mathrm{c}} = \mathbf{D}^*_{\mathrm{c}}$ and the notions of a c-dualizing complex and a t-dualizing complex coincide with the usual notion of a dualizing complex. We point out that according to [LNS, p. 106, Prop. 9.2.2], a t-dualizing complex lies in $\mathbf{D}^b_{\mathrm{qct}}(\mathscr{X})$.

Example 2.3.2. Let X be an ordinary scheme and $\kappa\colon \mathscr{X} \to X$ its completion along a closed subscheme Z. Then for any dualizing complex $\mathcal{R}$ on X, $\kappa^*\mathcal{R}$ is c-dualizing on $\mathscr{X}$ and $\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^*\mathcal{R} \cong \kappa^*\mathbf{R}\varGamma_Z\mathcal{R}$ is t-dualizing and lies in $\mathbf{D}^*_{\mathrm{c}}$ [AJL2, p. 25, 2.5.2(2)]. In particular, if k is a field and $\mathscr{X}$ is the formal spectrum of $A := k[[X_1, \dots, X_n]]$ equipped with the $\mathfrak{m}$-adic topology where $\mathfrak{m} = (X_1, \dots, X_n)$, (which implies that $\mathscr{X}$ consists of a single point) then a c-dualizing complex on $\mathscr{X}$ is given by A while a t-dualizing complex is given by the injective hull of $k = A/\mathfrak{m}$.

For a fixed t-dualizing complex $\mathscr{R}$ on $\mathscr{X}$ define the dualizing functor $\mathcal{D}_{\mathrm{t}} = \mathcal{D}_{\mathrm{t}}(\mathscr{R})\colon \mathbf{D} \to \mathbf{D}$ by

$$\mathcal{D}_{\mathrm{t}} := \mathbf{R}\mathscr{H}om^\bullet(-, \mathscr{R}).$$

If $\mathscr{R} \in \mathbf{D}^*_{\mathrm{c}}$—equivalently, if $\mathscr{X}$ has a c-dualizing complex—then according to [AJL2, p. 28, Prop. 2.5.8]

(1) $\mathscr{E} \in \mathbf{D}_{\mathrm{c}}^{*} \Leftrightarrow \mathcal{D}_{\mathrm{t}}\mathscr{E} \in \mathbf{D}_{\mathrm{c}}$.
(2) $\mathscr{F} \in \mathbf{D}_{\mathrm{c}} \Leftrightarrow \mathcal{D}_{\mathrm{t}}\mathscr{F} \in \mathbf{D}_{\mathrm{c}}^{*}$.
(3) If $\mathscr{F}$ is in either $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$ or $\mathbf{D}_{\mathrm{c}}^{*}(\mathscr{X})$, the natural map is an isomorphism:

$$\mathscr{F} \xrightarrow{\sim} \mathcal{D}_{\mathrm{t}}\mathcal{D}_{\mathrm{t}}\mathscr{F}. \tag{2.3.3}$$

The above facts can be summarized as follows:

Proposition 2.3.4. [AJL2, p. 28, Prop. 2.5.8] *Let $\mathscr{X}$ be a formal scheme with a t-dualizing complex $\mathscr{R} \in \mathbf{D}_{\mathrm{c}}^{*}(\mathscr{X})$. Then the functor $\mathcal{D}_{\mathrm{t}}$ induces, in either direction, an anti-equivalence of categories between $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$ and $\mathbf{D}_{\mathrm{c}}^{*}(\mathscr{X})$.*

Regarding $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ as a full subcategory of $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$, [LNS, p. 108, Cor. 9.3.2] characterizes the essential image of $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ in $\mathbf{D}_{\mathrm{c}}^{*}(\mathscr{X})$ under the above anti-equivalence, and this characterization underpins most of the results in this paper. We describe this in subsection Subsection 2.5

2.4. **Cohen-Macaulay and Cousin complexes.** Let $(\mathscr{X}, \Delta) \in \mathbb{F}_c$. We say that a complex $\mathscr{F} \in \mathbf{D}^{+}(\mathscr{X})$ is *Cohen-Macaulay* with respect to Δ (or Cohen-Macaulay on Δ) if, for any $x \in \mathscr{X}$, $\mathrm{H}^i_x(\mathscr{F}) = 0$ for $i \neq \Delta(x)$. Here $\mathrm{H}^i_x\mathscr{F}$ is defined to be (the "hyper local cohomology" module) $\mathrm{H}^i(\mathbf{R}\Gamma_x\mathscr{F})$ (sometimes denoted $\mathbb{H}^i_x(\mathscr{F})$). We (again) refer the reader to [LNS, pp. 36–44, §§ 3.2 and 3.3] for more elaborate definitions of Cousin complexes and Cohen-Macaulay complexes.

Cohen-Macaulay complexes and Cousin complexes are intimately related. In [Su, Theorem 3.9] Suominen shows (and this, in a more general form, is the main result of that paper) that the full subcategory of $\mathbf{D}(\mathscr{X})$ consisting of Cohen-Macaulay complexes on $(\mathscr{X}, \Delta)$ is equivalent to the full subcategory of $\mathbf{C}(\mathscr{X})$ consisting of Cousin complexes on $(\mathscr{X}, \Delta)$ via the restriction of the localization functor $Q\colon \mathbf{K}(\mathscr{X}) \to \mathbf{D}(\mathscr{X})$ to the category of Cousin complexes on $(\mathscr{X}, \Delta)$. The category of Δ Cousin complexes can be regarded as a full subcategory of $\mathbf{K}(\mathscr{X})$ since any two maps between Δ Cousin complexes which are homotopic to each other are actually equal.[1] A pseudo-inverse for Q restricted to Δ Cousin complexes is provided by the restriction of the Cousin functor $E_\Delta\colon \mathbf{D}^{+}(\mathscr{X}) \to \mathbf{C}(\mathscr{X})$ to Δ Cohen-Macaulay complexes. (See [LNS, Prop. 3.3.1, p. 42].)

For this paper, we are not interested in all Cohen-Macaulay or all Cousin complexes on $(\mathscr{X}, \Delta)$. In[LNS] and [S] the interest was often in Cohen-Macaulay (resp. Cousin) complexes in $\mathbf{D}_{\mathrm{qct}}(\mathscr{X})$. In this paper our interests are more special. We will concentrate on Cohen-Macaulay complexes in $\mathbf{D}_{\mathrm{c}}^{*}(\subset \mathbf{D}_{\mathrm{qct}})$. To that end we denote by $\mathbf{cm}(\mathscr{X}, \Delta)$ the full subcategory of $\mathbf{D}_{\mathrm{c}}^{*}(\mathscr{X})$ consisting of Cohen-Macaulay complexes on $(\mathscr{X}, \Delta)$ and we denote by $\mathbf{coz}_\Delta(\mathscr{X})$ the full subcategory of the category of Cousin complexes on $(\mathscr{X}, \Delta)$ which corresponds to $\mathbf{cm}(\mathscr{X}, \Delta)$ under Suominen's equivalence above. Note that any complex in $\mathbf{coz}_\Delta(\mathscr{X})$ consists of $\mathcal{A}_{\mathrm{qct}}$-modules, see the paragraph above Lemma 3.2.2 in [LNS, page 40].

In [S] we used the symbols CM^* and Coz^*_Δ for $\mathbf{cm}$ and $\mathbf{coz}_\Delta$ respectively.

2.5. **An anti-equivalence.** We are now in a position to identify the subcategory of $\mathbf{D}_{\mathrm{c}}^{*}(\mathscr{X})$ which corresponds to $\mathcal{A}_{\mathrm{c}}(\mathscr{X}) \subset \mathbf{D}_{\mathrm{c}}(\mathscr{X})$ under the anti-equivalence of Proposition 2.3.4. First, given a t-dualizing complex $\mathscr{R}$ on $\mathscr{X}$, one has an associated codimension function $\Delta_{\mathscr{R}}$ [LNS, p. 106, 9.2.2(ii)(b)]. Moreover, $\mathscr{R}$ *is*

[1]In fact the results of Suominen show that Cousin complexes on $(\mathscr{X}, \Delta)$ can be regarded as a full subcategory of $\mathbf{D}(\mathscr{X})$.

Cohen-Macaulay with respect to $\Delta_{\mathscr{R}}$ [LNS, Prop. 9.2.2(iii)(a)]. According to [LNS, p. 108, Prop. 9.3.1 and Cor. 9.3.2] we have:

Proposition 2.5.1. [LNS] *Let* $\mathscr{X}$ *be a formal scheme with a t-dualizing complex* $\mathscr{R} \in \mathbf{D}_{\mathrm{c}}^{*}(\mathscr{X})$. *Let* $\Delta = \Delta_{\mathscr{R}}$ *be the associated codimension function. Then the functor* $\mathcal{D}_{\mathrm{t}}$ *induces, in either direction, an anti-equivalence between* $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ *and* $\mathbf{cm}(\mathscr{X};\Delta)$. *Thus there exists a commutative diagram as follows, with* $\equiv$ *denoting equivalence of categories, the vertical arrows being the standard inclusions, and* C° *denoting the category opposite to the category* C*:*

$$\begin{tikzcd} \mathbf{D}_{\mathrm{c}}(\mathscr{X}) \arrow[r, "\equiv", "\mathcal{D}_{\mathrm{t}}"'] & \mathbf{D}_{\mathrm{c}}^{*}(\mathscr{X})^{\circ} \\ \mathcal{A}_{\mathrm{c}}(\mathscr{X}) \arrow[u] \arrow[r, "\equiv", "\mathcal{D}_{\mathrm{t}}"'] & \mathbf{cm}(\mathscr{X};\Delta)^{\circ} \arrow[u] \end{tikzcd}$$

Proposition 2.5.1 was first proved by Yekutieli and Zhang [YZ, Thm. 8.9] for ordinary schemes of finite type over noetherian finite dimensional regular rings.

2.6. **Residual complexes.** Since $\mathbf{cm}(\mathscr{X};\Delta)$ is equivalent to $\mathbf{coz}_{\Delta}(\mathscr{X})$, one can restate the anti-equivalence between $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ and $\mathbf{cm}(\mathscr{X};\Delta)$ in terms of $\mathbf{coz}_{\Delta}(\mathscr{X})$. The resulting anti-equivalence between $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ and $\mathbf{coz}_{\Delta}(\mathscr{X})$ can be stated entirely in terms of complexes, i.e., within $\mathbf{C}(\mathscr{X})$ rather than $\mathbf{D}(\mathscr{X})$. As a first step toward this, we discuss the notion of a residual complex on a formal scheme.

On an ordinary scheme, we refer to [Hrt, p. 304] for a definition of a residual complex. Following [LNS, p. 104, 9.1.1], by a residual complex on a formal scheme $\mathscr{X}$ we mean a complex $\mathscr{R}$ of $\mathcal{A}_{\mathrm{t}}$-modules such that there exists a defining ideal $\mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$ with the property that for any $n > 0$, $\mathscr{H}om^{\bullet}(\mathscr{O}_{\mathscr{X}}/\mathscr{I}^{n}, \mathscr{R})$ is residual on the ordinary scheme $(\mathscr{X}, \mathscr{O}_{\mathscr{X}}/\mathscr{I}^{n})$. (cf. [Ye], [LNS, §9, footnotes]) The residual complex $\mathscr{R}$ induces a codimension function $\Delta = \Delta_{\mathscr{R}}$ on $\mathscr{X}$, and $\mathscr{R}$ is Δ-Cousin consisting of $\mathcal{A}_{\mathrm{qct}}$-modules [LNS, p. 106, Prop. 9.2.2]. Moreover, if $\mathscr{X}$ admits a residual complex, then $\mathscr{X}$ is universally catenary (since the corresponding statement is true for ordinary schemes).

According to [LNS, Prop. 9.2.2(iii)], if $\mathscr{D}$ is t-dualizing and $\Delta = \Delta_{\mathscr{D}}$, then $\mathscr{R} := E_{\Delta}\mathscr{D}$ is a residual complex. Moreover, since $\mathscr{D}$ is Cohen-Macaulay on $(\mathscr{X}, \Delta)$, there is a canonical isomorphism between $\mathscr{D}$ and $Q\mathscr{R}$ (see [LNS, p. 42, 3.3.1 and 3.3.2]). Moreover, it is immediate from the definition of $\Delta_{\mathscr{D}}$ that $\Delta_{\mathscr{D}} = \Delta_{\mathscr{R}}$. Since the presence of a t-dualizing complex forces $\mathscr{X}$ to be of finite Krull dimension [LNS, p. 106, Prop. 9.2.2(ii)], $\mathscr{R}$ must be a bounded complex. Conversely, if $\mathscr{R}$ is a bounded residual complex, then $Q\mathscr{R}$ is t-dualizing [LNS, Prop. 9.2.2(ii) and (iii)].

We need a little more terminology which will facilitate discussions on Cousin complexes. As in [S], let $\mathbb{F}^{r}$ denote the full subcategory of $\mathbb{F}$ whose objects $\mathscr{X}$ admit a bounded residual complex $\mathscr{R}$ (necessarily a t-dualizing complex) such that $Q\mathscr{R} \in \mathbf{D}_{\mathrm{c}}^{*}(\mathscr{X})$. Note that the presence of such an $\mathscr{R}$ on $\mathscr{X} \in \mathbb{F}^{r}$ is equivalent to the existence of a c-dualizing complex on $\mathscr{X}$. Next consider the full subcategory $\mathbb{F}_{\mathrm{c}}^{r}$ of $\mathbb{F}_{c}$ consisting of pairs $(\mathscr{X}, \Delta) \in \mathbb{F}_{c}$ with $\mathscr{X} \in \mathbb{F}^{r}$. We remind the reader that a morphisms $(\mathscr{X}, \Delta') \to (\mathscr{Y}, \Delta)$ in $\mathbb{F}_{\mathrm{c}}^{r}$ is therefore a map $f\colon \mathscr{X} \to \mathscr{Y}$ in $\mathbb{F}^{r}$ and such that for $x \in \mathscr{X}$ and $y = f(x)$, $\Delta(y) - \Delta'(x)$ is equal to the transcendence degree of the residue field $k(x)$ of x over the residue field $k(y)$ of y. In other words,

if $f^\sharp\Delta$ is defined by the formula

$$f^\sharp\Delta(x) := \Delta(y) - \text{tr.deg.}_{k(y)}k(x) \qquad (x \in \mathscr{X}, y := f(x)) \tag{2.6.1}$$

then $\Delta' = f^\sharp\Delta$. One checks that $f^\sharp\Delta$ is always a codimension function on $\mathscr{X}$. If $(\mathscr{X}, \Delta) \in \mathbb{F}^r_c$ then a Cohen-Macaulay (resp. Cousin) complex on $(\mathscr{X}, \Delta)$ is a complex in $\mathbf{cm}(\mathscr{X}; \Delta)$ (resp. $\mathbf{coz}_\Delta(\mathscr{X})$).

2.7. **Cousin complexes and coherent sheaves.** As seen so far, for any $\mathscr{X}$ admitting a c-dualizing complex (equivalently, admitting a t-dualizing complex in $\mathbf{D}^*_{\mathrm{c}}$) with associated codimension function Δ, the category $\mathbf{cm}(\mathscr{X}, \Delta)$ is closely related to two abelian categories, namely, $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ via dualizing and $\mathbf{coz}_\Delta(\mathscr{X})$ via the cousin functor E_Δ. We now relate these two abelian categories directly.

Fix $(\mathscr{X}, \mathscr{R})$ with $\mathscr{R}$ a residual complex on the formal scheme $\mathscr{X}$ and set $\Delta = \Delta_{\mathscr{R}}$ and $\mathbf{coz}(\mathscr{X}) = \mathbf{coz}_\Delta(\mathscr{X})$. By [LNS, p. 104, Lemma 9.1.3] and [Hrt, p. 123], we see that $\mathscr{R}$ is a complex of $\mathcal{A}_{\mathrm{qct}}(\mathscr{X})$-injectives. Let $\mathscr{F}$ be a complex of $\mathcal{A}_{\vec{\mathrm{c}}}$-modules. (Recall that $\mathcal{A}_{\vec{\mathrm{c}}}$-modules are direct limits of coherent modules and there are inclusions $\mathcal{A}_{\mathrm{c}} \subset \mathcal{A}_{\vec{\mathrm{c}}}$, $\mathcal{A}_{\mathrm{qct}} \subset \mathcal{A}_{\vec{\mathrm{c}}}$.) For any such $\mathcal{F}$ we shall now make the identification

$$\mathcal{D}_{\mathrm{t}}\mathscr{F} = \mathscr{H}om^\bullet_{\mathscr{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{R}).$$

This can be justified using [AJL2, page 27, 2.5.6] (where the proof holds only when $\mathscr{F}$ is a complex of $\mathcal{A}_{\vec{\mathrm{c}}}$-modules and not for $\mathscr{F} \in \mathbf{D}_{\vec{\mathrm{c}}}$ as claimed). Using this version of $\mathcal{D}_{\mathrm{t}}$, we make the following three observations:

1) For $\mathscr{M} \in \mathcal{A}_{\mathrm{c}}(\mathscr{X})$, the complex

$$\mathscr{M}' := \mathscr{H}om^\bullet_{\mathscr{O}_{\mathscr{X}}}(\mathscr{M}, \mathscr{R})$$

lies in $\mathbf{coz}(\mathscr{X})$. That $\mathscr{M}'$ is a Cousin complex follows easily from the definitions but to see that its image in the derived category lies in $\mathbf{D}^*_{\mathrm{c}}$ we need Proposition 2.5.1, and the equivalence between $\mathbf{coz}(\mathscr{X})$ and $\mathbf{cm}(\mathscr{X}; \Delta)$.

2) For $\mathscr{F} \in \mathbf{coz}(\mathscr{X})$, the $\mathscr{O}_{\mathscr{X}}$-module

$$\mathscr{F}^* := \mathscr{H}om_{\mathbf{C}(\mathscr{X})}(\mathscr{F}, \mathscr{R})$$

lies in $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ and there is a natural quasi-isomorphism

$$\mathscr{F}^* \to \mathcal{D}_{\mathrm{t}}\mathscr{F}. \tag{2.7.1}$$

Indeed, note that for any object $\mathscr{G} \in \mathbf{coz}(\mathscr{X})$, we have

$$\mathscr{H}om(\mathscr{G}, \mathscr{R}) = H^0(\mathscr{H}om^\bullet(\mathscr{G}, \mathscr{R})) = H^0(\mathcal{D}_{\mathrm{t}}\mathscr{G}).$$

(To see the first equality, note that the only $\mathbf{C}(\mathscr{X})$-map $\mathscr{G} \to \mathscr{R}$ homotopic to zero is the zero map, for $\mathscr{G}$ and $\mathscr{R}$ are Δ-Cousin.) The assertions for $\mathscr{F}^*$ follow from Proposition 2.5.1 and the fact that $\mathbf{coz}(\mathscr{X})$ is equivalent to $\mathbf{cm}(\mathscr{X}; \Delta)$.

3) The operations $\mathscr{M} \mapsto \mathscr{M}'$ and $\mathscr{F} \mapsto \mathscr{F}^*$ are inverse operations. In greater detail:

(i) For $\mathscr{M} \in \mathcal{A}_{\mathrm{c}}(\mathscr{X})$, the natural map in $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ given by "evaluation" is an isomorphism

$$\mathscr{M} \xrightarrow{\ \sim\ } (\mathscr{M}')^*. \tag{2.7.2}$$

Indeed, in $\mathbf{D}_{\mathrm{c}}$, the above map is equivalent to (2.3.3).

(ii) For $\mathscr{F} \in \mathbf{coz}(\mathscr{X})$, the natural map in $\mathbf{coz}(\mathscr{X})$ given by "evaluation" is an isomorphism

$$\mathscr{F} \xrightarrow{\sim} (\mathscr{F}^*)'. \tag{2.7.3}$$

As in (i), this follows from (2.3.3) for objects in $\mathbf{D}_{\mathrm{c}}^*$.

Note that the correspondences $\mathscr{M} \mapsto \mathscr{M}'$ and $\mathscr{F} \mapsto \mathscr{F}^*$ are functorial, defining contravariant functors $-'$ and $-^*$. Here then is the restatement of Proposition 2.5.1:

Proposition 2.7.4. *The functors*

$$-^*\colon \mathbf{coz}(\mathscr{X}) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})^\circ$$

and

$$-'\colon \mathcal{A}_{\mathrm{c}}(\mathscr{X})^\circ \to \mathbf{coz}(\mathscr{X})$$

are pseudoinverses via (2.7.2) *and* (2.7.3)*, and therefore set up an anti-equivalence of categories between* $\mathbf{coz}(\mathscr{X})$ *and* $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$*.*

Remark 2.7.5. The functors $-'$ and $-^*$ depend upon the choice of $\mathscr{R}$ (as we will make explicit later in this remark). It will be clear from the context what the underlying residual complex is. There will be occasions when we deal with maps $f\colon (\mathscr{X}, \Delta') \to (\mathscr{Y}, \Delta)$ in $\mathbb{F}_{\mathrm{c}}^r$, with a residual complex $\mathscr{R}$ on $\mathscr{Y}$ and a residual complex $f^\sharp\mathscr{R}$ on $\mathscr{X}$, but even here it will be clear from the context, which residual complex is being used and when. As an example, for the symbol $(f^*\mathscr{F}^*)'$, it is to be assumed that the "upper star" occurring as a superscript of $\mathscr{F}$ is with respect to $\mathscr{R}$ and the "prime" outside the parenthesis is with respect to $f^\sharp\mathscr{R}$. As for the dependence on $\mathscr{R}$, if $F_{\mathscr{R}}\colon \mathcal{A}_{\mathrm{c}}(\mathscr{X})^\circ \to \mathbf{coz}(\mathscr{X})$ and $G_{\mathscr{R}}\colon \mathbf{coz}(\mathscr{X}) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})^\circ$ denote $\mathscr{H}om^\bullet(-,\mathscr{R})$ and $\mathscr{H}om(-,\mathscr{R})$ respectively, and $\mathscr{R}'$ is another residual complex whose associated codimension function is also Δ, then $F_{\mathscr{R}'} \cong F_{\mathscr{R}} \otimes \mathcal{L}$ and $G_{\mathscr{R}'} \cong G_{\mathscr{R}} \otimes \mathcal{L}$ where $\mathcal{L} = \mathscr{H}om(\mathscr{R}, \mathscr{R}')$. Note that $\mathcal{L}$ is an invertible $\mathscr{O}_{\mathscr{X}}$-module with inverse $\mathscr{H}om(\mathscr{R}', \mathscr{R})$ and we have the relation $\mathscr{R}' \cong \mathscr{R} \otimes \mathcal{L}$.

3. The S_2 condition

3.1. **The map $\mathsf{s}(\mathscr{G})$.** We first state a part of [LNS, p. 109, Prop. 9.3.5] in a form that is useful to us. The content of the Proposition is that the Cousin complex $E_\Delta(\mathscr{M})$ of a complex $\mathscr{M}$ depends *only on the zeroth cohomology of its dual* $\mathcal{D}_{\mathrm{t}}(\mathscr{M})$.

Proposition 3.1.1. [LNS] *Let* $\mathscr{X} \in \mathbb{F}^r$, $\mathscr{R}$ *a residual complex which is a* t*-dualizing complex in* $\mathbf{D}_{\mathrm{c}}^*\mathscr{X}$*, and let* $\Delta = \Delta_{\mathscr{R}}$*. If* $\theta\colon \mathscr{F} \to \mathscr{G}$ *is a map in* $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$ *such that* $H^m(\theta)\colon H^m(\mathscr{F}) \to H^m(\mathscr{G})$ *is an isomorphism, then the induced map*

$$E_{\Delta-m}(\mathcal{D}_{\mathrm{t}}\theta)\colon E_{\Delta-m}(\mathcal{D}_{\mathrm{t}}\mathscr{G}) \xrightarrow{\sim} E_{\Delta-m}(\mathcal{D}_{\mathrm{t}}\mathscr{F}).$$

is an isomorphism of Cousin complexes in $\mathbf{coz}_{\Delta-m}(\mathscr{X})$*. In particular, truncation on* $\mathscr{F}$ *induces natural isomorphisms*

$$E_\Delta(\mathcal{D}_{\mathrm{t}}\mathscr{F}) \cong E_\Delta(\mathcal{D}_{\mathrm{t}}(H^0\mathscr{F})) = E_\Delta((H^0\mathscr{F})') \cong (H^0\mathscr{F})'.$$

The proof of $E_{\Delta-m}(\mathcal{D}_{\mathrm{t}}\theta)$ being an isomorphism is contained in the opening paragraph of the proof of [LNS, p. 109, Prop. 9.3.5]. The fact that $E_{\Delta-m}(\mathcal{D}_{\mathrm{t}}\theta)$ is in $\mathbf{coz}_{\Delta-m}(\mathscr{X})$ follows from the last part of the statement of *loc.cit.* The last isomorphism of the above proposition holds because $(H^0\mathscr{F})'$ is already a Cousin complex.

We would like to define the notion of an S_2 module with respect to a codimension function. For this we need to recall certain parts of [LNS, pp. 108–111, §§ 9.3], especially as it relates to Corollary 9.3.6 of *loc.cit*. Let $(\mathscr{X}, \Delta)$ be an ordinary scheme in $\mathbb{F}_c^r$ and fix a residual complex $\mathscr{R}$ on $\mathscr{X}$ that is dualizing and with $\Delta_{\mathscr{R}} = \Delta$. Let $\mathscr{M} \in \mathcal{A}_c(\mathscr{X})$ and for simplicity, assume that

$$\min\{n \mid H^n\mathscr{M}' \neq 0\} = 0.$$

Let $\theta\colon H^0(\mathscr{M}') \to \mathscr{M}'$ be the obvious canonical map in $\mathbf{D}_c^+$ induced by truncation. Then θ induces a $\mathbf{D}(\mathscr{X})$ map (cf. [LNS, p. 109, 9.3.6])

$$\mathsf{s}(\mathscr{M})\colon \mathscr{M} \to E_\Delta(\mathscr{M}) \tag{3.1.2}$$

defined by the natural maps

$$\mathscr{M} \xrightarrow{\ \sim\ } \mathscr{M}'^{*} \xrightarrow{\mathcal{D}_{\mathsf{t}}(\theta)} (H^0(\mathscr{M}'))' \xrightarrow[3.1.1]{\cong} E_\Delta(\mathscr{M}).$$

In fact $\mathsf{s}(-)$ is defined for any $0 \neq \mathscr{G} \in \mathbf{D}_c^{*-}(\mathscr{X})$. We refer to [LNS, p. 109, Cor. 9.3.6] for more details.

3.2. **Coherent S_2 sheaves on ordinary schemes.** For the rest of this section, *all schemes are ordinary* and, as before, lie in $\mathbb{F}^r$, which translates—in this situation—to the existence of a dualizing complex on that scheme. We first recall the "classical" definitions on a commutative ring.

Fix a noetherian commutative ring A. Recall that a finite A-module M is said to satisfy Serre's condition (S_n) if $\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{n, \dim M_\mathfrak{p}\}$ for every prime ideal $\mathfrak{p}$ of A. According to [SS, p.516, Example 4.4], M satisfies condition (S_n) if and only if its Cousin complex $E_{\mathscr{H}}(M)$ with respect to the "height filtration of M" $\mathscr{H} = (H_i)$ with $H_i = \{\mathfrak{p} \in \operatorname{Supp}_A(M) \mid \operatorname{ht}_M(\mathfrak{p}) \geq 1\}$ is exact at its kth term for $1 \leq k \leq n-2$ and if the natural map $M \to H^0(E_{\mathscr{H}}(M))$ is an isomorphism[2] (see also [Sh3, p. 621, Theorem (2.2)]). In what follows, we prefer to use the symbol S_n without the traditional parenthesis for (S_n).

This motivates the following definition.

Definition 3.2.1. Let $(X, \Delta) \in \mathbb{F}_c^r$ and suppose $\mathscr{R}$ is a residual complex on (X, Δ). We say $\mathscr{M} \in \mathcal{A}(X)$ is an S_2-module on (X, Δ) (or S_2 on (X, Δ); or simply Δ-S_2) if

(a) $\mathscr{M} \in \mathcal{A}_c(X)$;
(b) $\min\{n \mid H^n(\mathscr{M}') \neq 0\} = 0$;
(c) With $\mathsf{s}(\mathscr{M})\colon \mathscr{M} \to E_\Delta(\mathscr{M})$ the map in (3.1.2), we have

$$H^0(\mathsf{s}(\mathscr{M}))\colon \mathscr{M} \xrightarrow{\ \sim\ } H^0E_\Delta\mathscr{M}.$$

Let the full subcategory of $\mathcal{A}_c(\mathscr{X})$ consisting of Δ-S_2 modules be denoted $S_2(\Delta)$. (Note: By definition, a Δ-S_2-module cannot be zero.)

The connection between the notions of Δ-S_2-modules and of S_2-modules is given in Proposition 3.2.4. But first we wish to make a couple of remarks.

[2] Strictly speaking, Sharp does not follow Hartshorne's recipe of constructing a Cousin complex of M for the height filtration $\mathscr{H}$. But Sharp's Cousin complexes also live on the skeletons induced $\mathscr{H}$ and hence by the uniqueness part of [Hrt, Page 232, IV Prop 2.3] and by [SS, pp. 500–501, § 1.1], these constructions coincide.

Remarks 3.2.2. 1) In spite of appearances, the Δ-S_2 condition does not depend on $\mathscr{R}$, but only on Δ. This is seen in two steps. First, if $\mathscr{S}$ is a second residual complex, with associated codimension function Δ, then $\mathscr{S} = \mathscr{R} \otimes \mathcal{L}$, with $\mathcal{L}$ an invertible sheaf on X. This means condition (b) above does not depend on $\mathscr{R}$. Second, the map $\mathsf{s}(\mathscr{M})$ is independent of $\mathscr{R}$, for it has the property that any map $\mathscr{M} \to \mathscr{F}$ with $\mathscr{F} \in \mathbf{cm}(X, \Delta)$ factors uniquely through $\mathsf{s}(\mathscr{M})$ (cf. [LNS, p. 109, 9.3.6(i)]).

2) If $\mathscr{M}$ satisfies (a) and (b) of Definition 3.2.1, then the $\mathbf{D}(X)$-map $\mathsf{s}(\mathcal{M})$ can be uniquely realized as a $\mathbf{C}(X)$-map. Indeed, since $\mathrm{H}^i_x\mathscr{M} = 0$ for $i < 0$, $E_\Delta\mathscr{M}$ has no non-zero components in negative degrees. We should point out that in this case, the map $\mathsf{s}(\mathscr{M})$ has a concrete realization, namely it is given by the natural localization map $\mathscr{M} \to \oplus_x i_x(\mathscr{M}_x)$, as x ranges over points with Δ value 0. This can be seen by base changing to the scheme $\mathrm{Spec}(\mathscr{O}_{X,x})$ where $\Delta(x) = 0$, and using the fact that $\mathsf{s}(\mathscr{M})$ behaves well with respect to localizations.

3) From the definition of $\mathsf{s}(\mathcal{M})$ we see that any $\mathscr{M}$ satisfying (a) and (b) of Definition 3.2.1 is Δ-S_2 iff the natural map $\mathscr{M} \to H^0((H^0(\mathscr{M}'))')$, induced by applying $H^0 \circ \mathcal{D}_{\mathrm{t}}$ to the natural truncation map $H^0(\mathscr{M}') \to \mathscr{M}'$, is an isomorphism.

We now give the connection between Δ-S_2 and S_2 modules. If A is a noetherian ring, we transplant notations and concepts for quasi-coherent sheaves on $X = \mathrm{Spec}(A)$ to modules on A in an obvious way, and the notations are self-explanatory. Thus if Δ is a codimension function on X, and M is an A-module, then $E_\Delta(M)$ means the complex of global sections of $E_\Delta(\mathscr{M})$ where $\mathscr{M}$ is the quasi-coherent $\mathscr{O}_X$-module corresponding to M. There are certain commutative algebra conventions we will follow whenever we move to that mode. Complexes of A-modules will have "bullets" as superscripts. A dualizing complex $D^\bullet$ of A-modules is a complex whose corresponding complex of quasi-coherent $\mathscr{O}_X$-modules is residual. If $D^\bullet$ is a dualizing complex, then the associated codimension function Δ is defined by the property that $\mathrm{H}^i_{\mathfrak{p}}(D^\bullet_{\mathfrak{p}}) = 0$ for $i \neq \Delta(\mathfrak{p})$ and $\mathrm{H}^{\Delta(\mathfrak{p})}_{\mathfrak{p}}(D^\bullet_{\mathfrak{p}}) = J_A(A/\mathfrak{p})$, where $J_A(A/\mathfrak{p})$ is the injective hull of the A-module $A/\mathfrak{p}$, and $D^\bullet_{\mathfrak{p}}$ means the localization $D^\bullet$ at $\mathfrak{p}$. Note that, by the conventions of commutative algebra $D^\bullet$ is Cousin with respect to its associated codimension function. Finally if Δ is a codimension function on $\mathrm{Spec}(A)$, and $C^\bullet$ a Cousin complex of A-modules with respect to Δ (i.e., $C^\bullet = E_\Delta(C^\bullet)$), then

$$C^\bullet(\mathfrak{p}) := H^{\Delta(\mathfrak{p})}(\Gamma_{\mathfrak{p}} C^\bullet_{\mathfrak{p}}) \qquad\qquad (\mathfrak{p} \in \mathrm{Spec}(A)).$$

Lemma 3.2.3. *Let A be a noetherian ring such that* $\mathrm{Spec}(A)$ *has a codimension function Δ and let M be a non-zero finitely generated A-module. Then* $\mathrm{ht}_M = \Delta|_{\mathrm{Supp}(M)}$ *if and only if* $\mathrm{Min}(M) = \{\mathfrak{p} \in \mathrm{Supp}(M) \mid \Delta(\mathfrak{p}) = 0\}$.

Proof. This is obvious. $\square$

Proposition 3.2.4. *Let A be a noetherian ring with a dualizing complex $D^\bullet$ whose associated codimension function is Δ. Then $M \in S_2(\Delta)$ if and only if M is S_2 and* $\mathrm{ht}_M = \Delta|_{\mathrm{Supp}(M)}$.

Proof. In what follows $\mathscr{H} = (H_i)$ is the filtration on $\mathrm{Spec}(A)$ given by $H_i = \{\mathfrak{p} \mid \mathrm{ht}_M(\mathfrak{p}) \geq i\}$. Suppose $M \in S_2(\Delta)$. The map $\mathsf{s} = \mathsf{s}(M)\colon M \to E_\Delta(M)$ of (3.1.2) is then given by $M \to \oplus_{\Delta(\mathfrak{p})=0} M_{\mathfrak{p}}$ (see 2) of Remarks 3.2.2), and $H^0(\mathsf{s})$ is an isomorphism. It follows that $\mathrm{Min}(M) = \{\mathfrak{p} \in \mathrm{Supp}(M) \mid \Delta(\mathfrak{p}) = 0\}$. Therefore by Lemma 3.2.3, $\mathrm{ht}_M = \Delta|_{\mathrm{Supp}(M)}$, whence the Cousin complex $E_{\mathscr{H}}(M)$ of M with

respect to the filtration $\mathscr{H}$ of M, is the Cousin complex $E_\Delta(M)$. Since $H^0(\mathsf{s})$ is an isomorphism, it follows that M is S_2 by [SS, p. 516, Example 4.4].

Conversely, suppose M is S_2 and $\mathrm{ht}_M = \Delta|_{\mathrm{Supp}(M)}$. Then $E_{\mathscr{H}}(M) = E_\Delta(M)$, and the natural map $M \to E_\Delta(M)$ (i.e., the one induced by $M \to \oplus_{\Delta(\mathfrak{p})=0} M_\mathfrak{p}$) yields an isomorphism on applying H^0. It therefore only remains to show that $H^0(\mathrm{Hom}^\bullet_A(M, D^\bullet)) \neq 0$. Now note that $\mathrm{Hom}^\bullet_A(M, D^\bullet)$ has no negative terms. Indeed, if $\Delta(\mathfrak{p}) < 0$, then $\mathfrak{p} \notin \mathrm{Supp}(M)$, and therefore we have $\mathrm{Hom}_A(M, D^\bullet(\mathfrak{p})) = \mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, D^\bullet(\mathfrak{p})) = 0$. Also note that for any $\mathfrak{p}$ such that $\Delta(\mathfrak{p}) = 0$ (e.g., by Lemma 3.2.3, if $\mathfrak{p} \in \mathrm{Min}(M)$), $D^\bullet_\mathfrak{p}$ has no positive terms and $D^0_\mathfrak{p} = D^\bullet(\mathfrak{p})$. It follows that for any such $\mathfrak{p}$ we have,

$$\mathrm{Hom}^\bullet_A(M, D^\bullet)_\mathfrak{p} = \mathrm{Hom}^\bullet_{A_\mathfrak{p}}(M_\mathfrak{p}, D^\bullet(\mathfrak{p})).$$

The last complex is a one term complex, concentrated at degree 0, and equal in that degree to the $A_\mathfrak{p}$-module $\mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, D^\bullet(\mathfrak{p}))$. Therefore, $H^0(\mathrm{Hom}^\bullet_A(M, D^\bullet))_\mathfrak{p} = \mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, D^\bullet(\mathfrak{p}))$, i.e., to the Matlis dual of the finitely generated $A_\mathfrak{p}$-module $M_\mathfrak{p}$, which is non-zero whenever $\mathfrak{p} \in \mathrm{Min}(M)$. □

Remark 3.2.5. Let A be a local ring such that $X = \mathrm{Spec}(A)$ possesses a dualizing complex (assumed residual, to keep with commutative algebraic conventions) and let $d = \dim A$. Suppose A has a canonical module K, i.e., a module such that $\widehat{K} = \mathrm{Hom}_A(\mathrm{H}^d_\mathfrak{m}(A), J_A(k(\mathfrak{m})))$, where $J_A(k(\mathfrak{m})$ is the injective hull of the A-module $k(\mathfrak{m})$. Let Δ_{F} be the *fundamental codimension function* on X, i.e., $\Delta_F(\mathfrak{p}) = d - \dim(A/\mathfrak{p})$. Let $D^\bullet$ be a dualizing complex with codimension function Δ_F. It is well known that $K \cong H^0(D^\bullet)$. For $M \in \{A, K\}$ one checks that M is S_2 if and only if M is (Δ_F)-S_2. Indeed if A is S_2, A is equidimensional, whence $\mathrm{ht}_A = \Delta_F$ (see [Ao, 1.9]). If K is S_2, then K is a submodule of D^0, whence $\mathrm{Min}(K) \subset \{\mathfrak{p} \mid \Delta_F(\mathfrak{p}) = 0\}$. On the other hand, $K_\mathfrak{p}$ is a canonical module for $A_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$, whence $\mathrm{Supp}(K) = \mathrm{Spec}(A)$. It follows that $\mathrm{Min}(K) = \{\mathfrak{p} \mid \Delta_F(\mathfrak{p}) = 0\}$, and from this it is easy to check that $\mathrm{ht}_K = \Delta_F$.

Lemma 3.2.6. *Let $\mathscr{M}$ be S_2 on (X, Δ) and $\mathscr{R}$ a residual complex with Δ as its associated codimension function. Then the Cousin complex $\mathscr{M}' \in \mathbf{coz}_\Delta(X)$ has no non-vanishing terms in negative degrees.*

Proof. Note that the first non-vanishing homology of a Cousin complex $\mathscr{C}$ (with respect to Δ) appears at the first non-zero term of $\mathscr{C}$. More precisely,

$$(*) \qquad \min\{n \mid H^n(\mathscr{C}) \neq 0\} = \min\{p \mid \mathscr{C}^p \neq 0\}.$$

To see this, first let $\mathscr{C}(x)$ $(x \in X)$ be as in [LNS, § 3.2, p. 37, para. 6], i.e., $\mathscr{C}(x)$ is an $\widehat{\mathscr{O}}_{X,x}$-module isomorphic to $\mathrm{H}^{\Delta(x)}_x(\mathscr{C})$ and $\mathscr{C}^p = \oplus_{\Delta(x)=p} i_x \mathscr{C}(x)$. Equality $(*)$ translates to:

$$\min\{n \mid H^n(\mathscr{C}) \neq 0\} = \min\{\Delta(x) \mid \mathscr{C}(x) \neq 0\}.$$

Now suppose x is a point with least Δ-value satisfying $\mathscr{C}(x) \neq 0$. Then $\mathscr{C}_x$ is a one-term complex (the term being $\mathscr{C}(x)$), whence

$$H^{\Delta(x)}(\mathscr{C})_x \cong H^{\Delta(x)}(\mathscr{C}_x) = \mathscr{C}(x) \neq 0$$

proving $(*)$. If $\mathscr{M}$ is S_2 with respect to Δ, then applying $(*)$ to $\mathscr{C} = \mathscr{M}'$, we get the result. □

Definition 3.2.7. Let $\mathscr{M}$ be S_2 on (X, Δ). We say $\mathscr{M}$ is *Cohen-Macaulay up to degree m on* (X, Δ) (or Δ-CM up to degree m) if $\mathsf{s}(\mathscr{M})_x \colon \mathscr{M}_x \to E_\Delta(\mathscr{M})_x$ is a quasi-isomorphism for every $x \in X$ with $\Delta(x) \le m$. The full subcategory of $S_2(\Delta)$ of modules which are Δ-CM up to degree m will be denoted $\mathbf{cm}(\Delta)_{\le m}$.

We remark that $\mathbf{cm}(\Delta)_{\le 2} = S_2(\Delta)$. This follows from the fact that for $p > 0$, the support of $H^p(E_\Delta(\mathscr{M}))$ has codimension at least $p+2$. At the other extreme, we use $\mathbf{cm}(\Delta)_{\le\infty}$ to denote the subcategory of all Δ-Cohen-Macaulay modules.

We are in a position to state and prove the first of our main theorems, namely, Theorem 3.2.8. We wish to make a few orienting remarks in order to understand the Theorem's relationship to the results of Dibaei and Tousi [DT, p. 19,Thm. 1.4] and of Kawasaki [Kw, Thm. 4.4] (see Remarks 3.2.9). Fix a residual complex $\mathscr{R}$ on (X, Δ). Let $\mathscr{M} \in S_2(\Delta)$ and $\mathscr{N} := (E_\Delta\mathscr{M})^*$. The Theorem is concerned with certain symmetric relations between $\mathscr{M}$ and $\mathscr{N}$. The first assertion is that $\mathscr{N} \in S_2(\Delta)$. According to the Theorem, stripped of its category theoretic language, the relations between $\mathscr{M}$ and $\mathscr{N}$ are as follows (where we write equalities for functorial isomorphisms to reduce clutter):

(i) $E_\Delta(\mathscr{N}) = \mathscr{M}'$; $E_\Delta(\mathscr{M}) = \mathscr{N}'$.
(ii) $\mathscr{N} = H^0(\mathscr{M}')$; $\mathscr{M} = H^0(\mathscr{N}')$.
(iii) $\mathscr{M} = E_\Delta(\mathscr{N})^*$ (note $\mathscr{N} := E_\Delta(\mathscr{M})^*$).
(iv) The following are equivalent:(a) $\mathscr{M} \in \mathbf{cm}(\Delta)_{\le m}$, (b) $\mathscr{N} \in \mathbf{cm}(\Delta)_{\le m}$, (c) $H^0(\mathscr{M}') \in \mathbf{cm}(\Delta)_{\le m}$, and (d) $H^0(\mathscr{N}') \in \mathbf{cm}(\Delta)_{\le m}$.

If $X = \operatorname{Spec}(A)$, and S_2 in the usual sense, then, in Remarks 3.2.9, we say more on the connections with the just cited results of Dibaei, Tousi [DT] and Kawasaki [Kw]. We would also like to draw the reader's attention to [LNS, p. 110, 9.3.7].

Finally some orienting remarks concerning the notations used in the theorem. Let E^*_Δ denote the composite $-^* \circ E_\Delta \colon \mathbf{D}^*_{\mathrm{c}}(\mathscr{X}) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})$, i.e., $E^*_\Delta(\mathscr{F}) = (E_\Delta(\mathscr{F}))^*$ for $\mathscr{F} \in \mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$. One may regard $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$, whence $S_2(\Delta)$, as a subcategory of $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$. By "restricting" E^*_Δ to $S_2(\Delta)$ we get a functor

$$T \colon S_2(\Delta) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})$$

given (at the level of objects) by $\mathscr{M} \mapsto (E_\Delta\mathscr{M})^* = E^*_\Delta(\mathscr{M})$. Similarly we have the functor $H^0 \circ (-') \colon \mathcal{A}_{\mathrm{c}}(\mathscr{X}) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})$. Restricting to $S_2(\Delta)$ we get

$$U \colon S_2(\Delta) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})$$

given at the level of objects by $\mathscr{M} \mapsto H^0(\mathscr{M}')$.

The theorem asserts (among other things) that T (resp. U) takes values in $S_2(\Delta)$. In such a case, by standard "extension" and "restriction" dualities, if $i \colon S_2(\Delta) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})$ is the inclusion functor, we get endofunctors $\mathbb{T}$ and $\mathbb{U}$ on $S_2(\Delta)$ defined uniquely by $i \circ \mathbb{T} = T$ and $i \circ \mathbb{U} = U$.

Theorem 3.2.8. *Let $\mathscr{R}$ be a residual complex on (X, Δ) and let $-^*$ and $-'$ be computed with respect to $\mathscr{R}$. Let $i \colon S_2(\Delta) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})$ be the natural embedding.*

(a) *The contravariant functors $T \colon S_2(\Delta) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})$ and $U \colon S_2(\Delta) \to \mathcal{A}_{\mathrm{c}}(\mathscr{X})$ take values in $S_2(\Delta)$.*
(b) *Let $\mathbb{T} \colon S_2(\Delta) \to S_2(\Delta)$ and $\mathbb{U} \colon S_2(\Delta) \to S_2(\Delta)$ be the contravariant functors defined by $i \circ \mathbb{T} = T$ and $i \circ \mathbb{U} = U$. Then*

$$\mathbb{T} \xrightarrow{\sim} \mathbb{U} \tag{3.2.8.1}$$

or, equivalently,

$$T \xrightarrow{\sim} U. \tag{3.2.8.2}$$

(c) *The contravariant functor* $\mathbb{T}$ *(and therefore* $\mathbb{U}$*) is an anti-equivalence of categories and is its own pseudo-inverse, i.e. ,*

$$\mathbb{T}^2 \cong \mathbf{1} \cong \mathbb{U}^2 \tag{3.2.8.3}$$

(d) *There is a functorial isomorphism*

$$E_\Delta \circ T \xrightarrow{\sim} \boldsymbol{-'}|_{S_2(\Delta)} \tag{3.2.8.4}$$

such that the following diagram commutes:

$$\begin{tikzcd}
(\boldsymbol{-'})^*|_{S_2(\Delta)} \arrow[rr, "\sim"', "(2.7.2)"] & & i \\
(E_\Delta T)^* \arrow[u, "(3.2.8.4)^*", "\wr"'] \arrow[rr, equal] & & i\mathbb{T}^2 \arrow[u, "\wr", "(3.2.8.3)"']
\end{tikzcd}$$

(Note that therefore (3.2.8.4) *and* (3.2.8.3) *determine each other.)*

(e) $\mathbb{T}\mathscr{M} \in \mathbf{cm}(\Delta)_{\leq m} \iff \mathscr{M} \in \mathbf{cm}(\Delta)_{\leq m} \iff \mathbb{U}\mathscr{M} \in \mathbf{cm}(\Delta)_{\leq m}$.

Remarks 3.2.9. For the more commutative algebraic minded readers, here are the re-interpretations in terms of rings and complexes of modules. As before, the notations used are the obvious transplants of the notations we have used for schemes and quasi-coherent sheaves, and are self-explanatory. In what follows A is a local ring of dimension d, possessing a dualizing complex $D^\bullet$, and $M \neq 0$ is an A-module.. Let $\Delta\colon \operatorname{Spec}(A) \to \mathbb{Z}$ be the associated codimension function. We remind the reader that it is standard in commutative algebra to assume that $D^\bullet$ is Cousin with respect to Δ (or, equivalently, $D^\bullet$ is residual). This implies

$$D^i = \bigoplus_{\substack{\mathfrak{p}\in\operatorname{Spec}(A),\\ \Delta(\mathfrak{p})=i}} J_A(A/\mathfrak{p}),$$

where, $J_A(A/\mathfrak{p})$ is the injective hull of the A-module $A/\mathfrak{p}$. The presence of a codimension function Δ on $\operatorname{Spec}(A)$ implies the presence of a *fundamental codimension function* Δ_F given by $(\mathfrak{p}) \mapsto d - \dim A/\mathfrak{p}$. If the codimension function of the $D^\bullet$ is Δ_F, then $D^\bullet$ is called a *fundamental dualizing complex* [Db, p.120, 1.1]. We denote the fundamental dualizing complex $D^\bullet_{F}$.

1) Suppose M is S_2 with respect to Δ (not necessarily equal to Δ_{F}). Theorem 3.2.8 asserts the existence of another finitely generated module, also S_2 with respect to Δ, in some fundamental sense, *dual* to M. This module can be described in two ways (and the fact that two descriptions are the "same" is one of the statements of Theorem 3.2.8). The first way is as follows: Consider the complex $T^\bullet = \operatorname{Hom}^\bullet_A(E_\Delta(M), D^\bullet)$ and set $T(M) = H^0(T^\bullet)$. The second way is to consider $U^\bullet = \operatorname{Hom}^\bullet_A(M, D^\bullet)$ and set $U(M) = H^0(U^\bullet)$. Then $T(M)$ and $U(M)$ are (functorially) isomorphic. In what follows, call the common module N.

2) The module M can be recovered from N as either $T(N)$ or as $U(N)$. This establishes a "symmetry" between M and N.

3) By [LNS, p.109, Prop. 9.3.5] the Cousin complexes $E_\Delta(M)$ and $E_\Delta(N)$ have finitely generated cohomology modules. (See also [DT, Thm. 3.2].) In fact $E_\Delta(M)$ can be identified with $\operatorname{Hom}^\bullet_A(N, D^\bullet)$ and $E_\Delta(N)$ can be identified with the complex $\operatorname{Hom}^\bullet_A(M, D^\bullet)$.

4)The above can be summarized by the diagram (1.1.1) which we reproduce (reminding the reader that coz^2_Δ is the essential image of $S_2(\Delta)$ under E_Δ):

$$\begin{tikzcd} S_2(\Delta) \arrow[r, leftrightarrow, "\text{dualize}"] \arrow[d, shift left, "\downarrow E_\Delta"] & \mathrm{coz}^2_\Delta \arrow[d, shift left, "\downarrow H^0"] \\ \mathrm{coz}^2_\Delta \arrow[u, shift left, "H^0\uparrow"] \arrow[r, leftrightarrow, "\text{dualize}"'] & S_2(\Delta) \arrow[u, shift left, "E_\Delta\uparrow"] \end{tikzcd}$$

Note that from the diagram, completing a clockwise circuit starting from either the northwest corner, or the the southeast corner amounts to saying $U(U(M)) \cong M$ for $M \in S_2(\Delta)$; and completing a counter-clockwise circuit starting from the same two vertices, amounts to saying $T(T(M)) \cong M$ for $M \in S_2(\Delta)$. If we imagine M as an object in the northwest corner, then its "dual" N occurs in the southeast corner (by following the transformations of M along any of the routes possible). At the southwest corner, we have the transform of M along the immediate the south pointing arrow and the transform of N along the immediate west pointing direction. The "equality" of these transforms amounts to the "equality" $E_\Delta(M) = \mathrm{Hom}^\bullet_A(N, D^\bullet)$. We leave it to the reader to use the diagram to work out other possible relations between M, N, $E_\Delta(M)$, and $E_\Delta(N)$.

5) The relationship between Theorem 3.2.8 and [DT, p. 19, Thm. 1.4] is complicated and needs to be explored in greater detail than we do in this paper. Here is what we can say. Let $k = \min\{j \mid \mathrm{Supp}_A(M) \cap \mathrm{Ass}_A(D^j_F) \neq 0\}$, and let $\Delta = \Delta_M = \Delta_F - k$. Define $U^\bullet$ and $U(M)$ as we did in 1), but without assuming M is Δ-S_2. The re-interpretation of [DT, Thm. 1.4] in our language is then as follows. In *loc. cit.* it is shown that there is a natural (and unique) map of complexes $E_\Delta(U(M)) \to \mathrm{Hom}^\bullet(M, D^\bullet)$ which lifts the identity on $U(M)$. Consider the condition

$$(\dagger) \qquad \mathrm{Min}(M) = \mathrm{Ass}(U(M)).$$

Condition (†) is readily seen to be equivalent to our preferred condition $\mathrm{ht}_M = \Delta|_{\mathrm{Supp}(M)}$ (see *loc. cit.* (ii)). Thus, according to Proposition 3.2.4, M is Δ-S_2 if and only if M is S_2 and (†) holds. Under the assumption that M is S_2 and (†) holds, the identification

$$E_\Delta(U(M)) = \mathrm{Hom}^\bullet_A(M, D^\bullet)$$

of 3) above is established by different methods in [DT, p. 19,Thm. 1.4](iv). However, the symmetric relationship between M and $U(M)$ ($= K^I_M$ in the notation of *loc. cit.*) is not fully explored, nor the fact that $U(M)$ is also $T(M)$. It should be pointed out that there are other results (concerning M and $U(M)$) in *loc. cit.* that are of considerable interest (see also [Kw, Thm. 4.4]).

6) As noted in Remark 3.2.5, A (resp. K) is S_2 if and only if it is Δ_F-S_2. Since $M = A$ (resp. $M = K$) yields $N = K$ (resp. $N = A$), it follows that A is S_2 if and only if K is S_2. In this case $E_{\Delta_F} = D^\bullet_F$ and $E_{\Delta_F}(A) = \mathrm{Hom}^\bullet_A(K, D^\bullet_F)$ (cf. [LNS, p.110, Example 9.3.7] and [Db, p.126, Thm. 4.6]). This observation has consequences later, when we generalize [Db, p. 125, Thm. 3.3] (see Remark 6.2.7).

7) As a matter of record we point out the well known fact that if A is S_2 then A has no embedded primes and is equidimensional. In particular, the fundamental codimension function in this case coincides with the height function on Spec$A$, $\mathfrak{p} \mapsto \mathrm{ht}(\mathfrak{p})$ (see, e.g., [DT, p.23, Rmk. 2.1]).

Proof of 3.2.8. Let $\mathscr{M} \in S_2(\Delta)$. By Proposition 3.1.1, there is a natural isomorphism $E_\Delta\mathscr{M} \cong (H^0(\mathscr{M}'))'$ and hence by dualizing we obtain an isomorphism

$$T\mathscr{M} = (E_\Delta\mathscr{M})^* \cong H^0(\mathscr{M}') = U(\mathscr{M})$$

which we take as our choice for (3.2.8.2). (Recall that since we are on an ordinary scheme, we have $\mathbf{D}_\mathrm{c}^* = \mathbf{D}_\mathrm{c}$.)

Next we claim that the natural map $U(\mathscr{M}) = H^0(\mathscr{M}') \to \mathscr{M}'$ satisfies the universal property of $\mathsf{s}(U(\mathscr{M}))$ as mentioned in 1) of Remarks 3.2.2. Suppose $\mathscr{F}$ is Cohen-Macaulay and $\alpha\colon H^0\mathscr{M}' \to \mathscr{F}$ a map of complexes. Upon dualizing we get the following natural maps.

$$\begin{tikzcd} \mathscr{M} \arrow[d, "\mathsf{s}(\mathscr{M})"'] & & \mathcal{D}_\mathrm{t}\mathscr{F} \arrow[d, "\mathcal{D}_\mathrm{t}(\alpha)"] \\ E\mathscr{M} & & (U(\mathscr{M}))' \arrow[ll, "\cong"'] \end{tikzcd}$$

The bottom row is an isomorphism of Cousin complexes while the top row consists of objects in $\mathcal{A}_\mathrm{c}$. Therefore, if $\mathscr{M}$ is Δ-S_2, so that $\mathscr{M} = H^0E\mathscr{M}$, then there exists a unique map $\mathcal{D}_\mathrm{t}\mathscr{F} \to \mathscr{M}$ that makes the above diagram commute. Thus, dualizing back we obtain our claim.

It follows from the universal property of $\mathsf{s}(U(\mathscr{M}))$ that there is a unique isomorphism $E(U(\mathscr{M})) \xrightarrow{\sim} \mathscr{M}'$ over $U(\mathscr{M})$. This shows that $U(\mathscr{M})$ is also Δ-S_2 and moreover via $U \xrightarrow{\sim} T$ we also get a choice for the isomorphism (3.2.8.4) of part (d). (Note that E of a module has no negative terms and hence $\mathscr{M}'$ satisfies 3.2.1(b).) Thus we obtain parts (a)–(d) of the theorem.

It remains to prove (e). In view of part (c), it suffices to prove any one of the implications in (e). We shall prove that $\mathscr{M} \in \mathbf{cm}(\Delta)_{\le m} \Longrightarrow \mathbb{T}\mathscr{M} \in \mathbf{cm}(\Delta)_{\le m}$.

We first remark that if $0 \neq \mathscr{N} \in \mathbf{cm}(\Delta)_{\le\infty}$, i.e., $\mathscr{N}$ is Cohen-Macaulay, then so is $\mathbb{U}\mathscr{N} \cong \mathbb{T}\mathscr{N}$. Indeed, by 2.5.1, the Cousin complex $\mathscr{N}'$ has only one non-vanishing homology and hence $\mathbb{U}\mathscr{N}$ is $\mathbf{D}$-isomorphic to $\mathscr{N}'$ which is Cohen-Macaulay.

Let $\mathscr{M} \in \mathcal{A}_\mathrm{c}$. Pick a point $x \in X$ and set $X'_x = \mathrm{Spec}\mathscr{O}_{X,x}$. Let $f_x\colon X'_x \to X$ be the canonical flat map. The constructions of E and s behave well with base-change to X_x. Since Cohen-Macaulayness of a module is equivalent to s being a quasi-isomorphism, one checks that $\mathscr{M} \in \mathbf{cm}(X,\Delta)_{\le m}$ iff for all $x \in X$ such that $\Delta(x) \le m$, it holds that $f_x^*\mathscr{M}$ is Cohen-Macaulay on X_x. Also note that f_x^* "commutes" with the functors s and $\mathbb{T}$ and for the corresponding notions on X_x we use the symbols s_x, $\mathbb{T}_x$ respectively.

Thus we have

$$\begin{aligned} \mathscr{M} \in \mathbf{cm}(X,\Delta)_{\le m} &\iff f_x^*\mathscr{M} \text{ is CM on } X_x \text{ for all } x \in X, \Delta(x) \le m \\ &\iff \mathbb{T}_x f_x^*\mathscr{M} \text{ is CM on } X_x \text{ for all } x \in X, \Delta(x) \le m \\ &\iff f_x^*\mathbb{T}\mathscr{M} \text{ is CM on } X_x \text{ for all } x \in X, \Delta(x) \le m \\ &\iff \mathbb{T}\mathscr{M} \in \mathbf{cm}(X,\Delta)_{\le m}. \end{aligned}$$

□

4. Connections with Grothendieck duality

In this section we use Proposition 2.5.1 (or, equivalently, Proposition 2.7.4) to extend and make transparent some of the results in [S], e.g. the result that a map

f is flat if and only if $f^!$ transforms Cohen-Macaulay complexes to appropriate Cohen-Macaulay complexes (cf. [S, Theorems 7.2.2 and 9.3.12]).

4.1. **Pull back of Cousin complexes.** We fix, for the rest of this discussion, a map $f\colon (\mathscr{X}, \Delta') \to (\mathscr{Y}, \Delta)$ in $\mathbb{F}_{\mathrm{c}}^r$ and a residual complex $\mathscr{R}$ on $(\mathscr{Y}, \Delta)$. Let Coz_Δ (resp. $\mathrm{Coz}_{\Delta'}$) represents the category of all Cousin complexes in $\mathbf{D}_{\mathrm{qct}}(\mathscr{Y})$ (resp. $\mathbf{D}_{\mathrm{qct}}(\mathscr{X})$). This category is larger than $\mathbf{coz}_\Delta$, the category Cousin complexes in $\mathbf{D}_{\mathrm{c}}^*(\mathscr{Y})$. Let

$$f^\sharp\colon \mathrm{Coz}_\Delta(\mathscr{Y}) \to \mathrm{Coz}_{\Delta'}(\mathscr{X}) \tag{4.1.1}$$

be the functor constructed in [LNS, p. 10, Main Theorem]. Very briefly, if $f^\sharp\Delta$ is the codimension function in (2.6.1), with f is smooth and d its relative dimension[3], then $f^\sharp(\mathscr{F})$ is isomorphic to the Cousin complex $E_{f^\sharp\Delta}(\mathbf{R}\varGamma'_{\mathscr{X}}(\mathbf{L}f^*\mathscr{F} \otimes_{\mathscr{O}_\mathscr{X}} \Omega^d_f[d]))$. If f is a closed immersion, then $f^\sharp(\mathscr{F})$ is isomorphic to the $f^\sharp\Delta$-Cousin complex $\mathscr{H}om^\bullet(\mathscr{O}_\mathscr{X}, \mathscr{F})$. A general $f \in \mathbb{F}_{\mathrm{c}}^r$, can locally be written as a closed immersion followed by a smooth map, giving an idea how one might construct $f^\sharp\mathscr{F}$ in this case. One has to show the results are independent of factorizations (into closed immersions followed by smooth maps), and have 2-functorial properties (i.e. $(gf)^\sharp \cong f^\sharp g^\sharp$ plus "associativity"), and this is what is done in [LNS]. In [S] it is shown that $f^\sharp(\mathscr{F})$ is a concrete approximation of the twisted inverse image $f^!\mathscr{F}$ of Alonso, Jeremías and Lipman [AJL2]. In the event f is pseudo-proper, $f^\sharp(\mathscr{F})$ represents the functor $\mathscr{G} \mapsto \mathrm{Hom}_{\mathbf{C}(\mathscr{Y})}(f_*\mathscr{G}, \mathscr{F})$ on $\mathrm{Coz}_{\Delta'}$ (see [S, p.185, Thm. 8.1.10]), and if f is an open immersion then $f^\sharp = f^*$.

For $\mathscr{F}$ in $\mathbf{coz}_\Delta(\mathscr{Y})$, define

$$f^{(\sharp)}_{\mathscr{R}}(\mathscr{F}) := \mathscr{H}om^\bullet_{\mathscr{X}}(f^*\mathscr{F}^*, f^\sharp\mathscr{R}) = (f^*\mathscr{F}^*)' \tag{4.1.2}$$

where "upper star" is with respect to $\mathscr{R}$ and $-'$ is with respect to $f^\sharp\mathscr{R}$. Since $\mathscr{F}^*$ is a coherent $\mathscr{O}_\mathscr{Y}$-module by Proposition 2.7.4, $f^{(\sharp)}_{\mathscr{R}}\mathscr{F}$ is in $\mathbf{coz}_{\Delta'}(\mathscr{X})$. Thus we have a functor

$$f^{(\sharp)}_{\mathscr{R}}\colon \mathbf{coz}_\Delta(\mathscr{Y}) \to \mathbf{coz}_{\Delta'}(\mathscr{X}).$$

The functor $f^{(\sharp)}_{\mathscr{R}}$ makes transparent many of the relationships established between the twisted inverse image functor $f^!$ and $f^\sharp$ in [S]. We will show in Theorem 5.3.3 that $f^{(\sharp)}_{\mathscr{R}}$ is essentially $f^\sharp|_{\mathbf{coz}}$. But first, we would like to show that $f^{(\sharp)}_{\mathscr{R}}$ is independent of $\mathscr{R}$.

Proposition 4.1.3. *Let f be as above. There is a family of isomorphisms*

$$\psi_{\mathscr{R},\mathscr{R}'} = \psi_{f,\mathscr{R},\mathscr{R}'}\colon f^{(\sharp)}_{\mathscr{R}'} \xrightarrow{\ \sim\ } f^{(\sharp)}_{\mathscr{R}}, \tag{4.1.3.1}$$

one for each pair of residual complexes $\mathscr{R}, \mathscr{R}'$ on $(\mathscr{Y}, \Delta)$ such that $\psi_{\mathscr{R},\mathscr{R}'} \circ \psi_{\mathscr{R}',\mathscr{R}''} = \psi_{\mathscr{R},\mathscr{R}''}$ (cocycle condition) for any three residual complexes $\mathscr{R}$, $\mathscr{R}'$, $\mathscr{R}''$ on $(\mathscr{Y}, \Delta)$.

Proof. The proof rests on the fact that there are isomorphisms between $\mathscr{R}'$ and $\mathscr{S} := \mathscr{R} \otimes \mathcal{L}$, where $\mathcal{L}$ is the coherent invertible $\mathscr{O}_\mathscr{Y}$-module $\mathscr{H}om(\mathscr{R}, \mathscr{R}')$, and that isomorphisms between $\mathscr{R}'$ and $\mathscr{S}$ are indexed by units in the ring $\Gamma(\mathscr{Y}, \mathscr{O}_\mathscr{Y})$ (since $\mathrm{Hom}(\mathscr{R}', \mathscr{R}') = \Gamma(\mathscr{Y}, \mathscr{O}_\mathscr{Y})$).

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[3]In affine terms, this means, if $(A, I) \to (B, J)$ is a pseudo-finite map of adic rings, then the map is formally smooth—in the sense of lifting idempotents—and for any prime ideal $\mathfrak{p} \subset A$, and $\mathfrak{q} \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$, the integer d is given by $d = \dim B_\mathfrak{q}/\mathfrak{p}B + \mathrm{tr.deg.}_{k(\mathfrak{p})k(\mathfrak{q})}$ [LNS, p.22, Def. 2.4.2 and p.28, Def. 2.6.2].

We first make the identification
$$f^{(\sharp)}_{\mathscr{R}} = f^{(\sharp)}_{\mathscr{S}}$$
via the canonical identifications $\mathscr{H}om(\mathscr{F},\mathscr{S}) = \mathscr{H}om(\mathscr{F},\mathscr{R})\otimes\mathcal{L}$, $f^\sharp\mathscr{S} = f^\sharp\mathscr{R}\otimes f^*\mathcal{L}$, and $\mathscr{H}om^\bullet(\mathscr{M}\otimes f^*\mathcal{L}, f^\sharp\mathscr{R}\otimes f^*\mathcal{L}) = \mathscr{H}om^\bullet(\mathscr{M},\, f^\sharp\mathscr{R})$ for a coherent sheaf $\mathscr{F}$ on $\mathscr{Y}$ and a coherent sheaf $\mathscr{M}$ on $\mathscr{X}$.

Next, pick an isomorphism $\alpha\colon \mathscr{R}' \xrightarrow{\ \sim\ } \mathscr{S}$. Then α induces an isomorphism
$$\psi_\alpha\colon f^{(\sharp)}_{\mathscr{R}'} \xrightarrow{\ \sim\ } f^{(\sharp)}_{\mathscr{S}} (= f^{(\sharp)}_{\mathscr{R}}).$$
In greater detail, $\psi_\alpha = q_\alpha^{-1}p_\alpha = s_\alpha r_\alpha^{-1}$ where $p_\alpha, q_\alpha, r_\alpha, s_\alpha$ are the maps induced by α in the commutative diagram below (where $\mathscr{H}om^\bullet = \mathscr{H}om^\bullet_{\mathscr{X}}$):
$$\begin{tikzcd}
\mathscr{H}om^\bullet(f^*\mathscr{H}om_{\mathscr{Y}}(\mathscr{F},\mathscr{R}'), f^\sharp\mathscr{R}') \arrow[r, "\sim", "p_\alpha"'] & \mathscr{H}om^\bullet(f^*\mathscr{H}om_{\mathscr{Y}}(\mathscr{F},\mathscr{R}'), f^\sharp\mathscr{S}) \\
\mathscr{H}om^\bullet(f^*\mathscr{H}om_{\mathscr{Y}}(\mathscr{F},\mathscr{S}), f^\sharp\mathscr{R}') \arrow[u, "r_\alpha", "\wr"'] \arrow[r, "s_\alpha"'] & \mathscr{H}om^\bullet(f^*\mathscr{H}om_{\mathscr{Y}}(\mathscr{F},\mathscr{S}), f^\sharp\mathscr{S}) \arrow[u, "\wr", "q_\alpha"']
\end{tikzcd}$$
Suppose $\beta\colon \mathscr{R}' \xrightarrow{\ \sim\ } \mathscr{S}$ is another isomorphism. We claim that $\psi_\alpha = \psi_\beta$. Note that there exists a (unique) unit $a \in \Gamma(\mathscr{Y},\, \mathscr{O}_{\mathscr{Y}})$ such that $\alpha = a\beta$, so that $p_\alpha = ap_\beta$ and $q_\alpha = aq_\beta$. It follows that $q_\alpha^{-1}p_\alpha = q_\beta^{-1}p_\beta$. This proves the claim. Setting $\psi_{\mathscr{R},\mathscr{R}'}$ equal to ψ_α, it is not difficult to establish the cocycle rules. $\square$

From the proposition we deduce a well defined functor

(4.1.3.2) $$f^{(\sharp)}\colon \mathbf{coz}_\Delta(\mathscr{Y}) \to \mathbf{coz}_{\Delta'}(\mathscr{X})$$

independent of $\mathscr{R}$, together with isomorphisms

(4.1.3.3) $$\sigma_{\mathscr{R}}: f^{(\sharp)}_{\mathscr{R}} \xrightarrow{\ \sim\ } f^{(\sharp)}$$

such that $\sigma_{\mathscr{R}}^{-1}\circ\sigma_{\mathscr{R}'} = \psi_{\mathscr{R},\mathscr{R}'}$.

4.2. **Grothendieck duality.** For f and $\mathscr{R}$ as above, in [S, § 9], functors $f^{(!)}_{\mathscr{R}}$ and $f^{(!)}$ are constructed,[4] more or less along the lines that $f^{(\sharp)}_{\mathscr{R}}$ and from it $f^{(\sharp)}$ are constructed. In slightly greater detail, if $\mathscr{F}$ is an object in $\mathbf{D}^*_{\mathrm{c}}(\mathscr{Y})\cap\mathbf{D}^+(\mathscr{Y})$, then
$$f^{(!)}_{\mathscr{R}}\mathscr{F} := \mathcal{D}'_{\mathrm{t}}\circ\mathbf{L}f^*\circ\mathcal{D}_{\mathrm{t}}(\mathscr{F})$$
where $\mathcal{D}_{\mathrm{t}}$ (resp. $\mathcal{D}'_{\mathrm{t}}$) is the dualizing functor in (2.3.3) associated to $\mathscr{R}$ (resp. $f^\sharp\mathscr{R}$)[5]. It is not hard to see that $f^{(!)}_{\mathscr{R}}$ is an object in $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})\cap\mathbf{D}^+(\mathscr{X})$ (see [S, §§ 9.2, p. 187], especially the discussion after (9.2.1)). The passage from $f^{(!)}_{\mathscr{R}}\mathscr{F}$ to $f^{(!)}\mathscr{F}$ is identical to the passage from $f^{(\sharp)}_{\mathscr{R}}$ to $f^{(\sharp)}$, and one has functorial isomorphisms
$$\theta_{\mathscr{R}} = \theta_{f,\mathscr{R}}\colon f^{(!)}_{\mathscr{R}} \xrightarrow{\ \sim\ } f^{(!)}$$
such that, for a second residual complex $\mathscr{R}'$ on $(\mathscr{Y},\, \Delta)$,
$$\phi_{\mathscr{R},\mathscr{R}'}(= \phi_{f,\mathscr{R},\mathscr{R}'}) := \theta_{\mathscr{R}}^{-1}\theta_{\mathscr{R}'} : f^{(!)}_{\mathscr{R}'} \xrightarrow{\ \sim\ } f^{(!)}_{\mathscr{R}}$$
satisfies cocycle rules.

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[4]more precisely $|f|^{(!)}_{\mathscr{R}}$ and $|f|^{(!)}$ are constructed, where $|f|\colon \mathscr{X} \to \mathscr{Y}$ is the map underlying $f\colon (\mathscr{X},\Delta') \to (\mathscr{Y},\Delta)$.

[5]$f^\sharp\mathscr{R}$ is also residual [LNS, p. 105, Prop. 9.1.4].

A couple of minor irritants need to be quickly addressed. In [S, § 9], the source and target of $f^{(!)}_{\mathscr{R}}$ and $f^{(!)}$ are complicated subcategories of $\mathbf{D}(\mathscr{Y})$ and $\mathbf{D}(\mathscr{X})$ respectively. For our purposes, it suffices to observe that the source contains $\mathbf{D}^*_{\mathrm{c}}(\mathscr{Y})\cap\mathbf{D}^+(\mathscr{Y})$. Thus in this paper, we regard $f^{(!)}_{\mathscr{R}}$ and $f^{(!)}$ as functors with source $\mathbf{D}^*_{\mathrm{c}}(\mathscr{Y})\cap\mathbf{D}^+(\mathscr{Y})$ and target $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})\cap\mathbf{D}^+(\mathscr{X})$:

$$f^{(!)}_{\mathscr{R}}\cong f^{(!)}\colon \mathbf{D}^*_{\mathrm{c}}(\mathscr{Y})\cap\mathbf{D}^+(\mathscr{Y})\to\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})\cap\mathbf{D}^+(\mathscr{X}).$$

A second point needs to be made. As in [S], we reserve the notation $f^!$ (as opposed to $f^{(!)}$) for the twisted inverse image functor obtained in [AJL2] (cf. [*Ibid,* p. 2, Thm. 2 and beginning of §§ 1.3]) for pseudo-proper maps, and extended to composites of compactifiable maps in [Nay, p. 261, 7.1.3]. We point out that $f^!$ and $f^{(!)}$ are canonically isomorphic when both are defined [S, p. 190, Thm. 9.3.10].

4.3. **Cousin complexes and duality.** Let f, $\mathscr{R}$, $\mathcal{D}_{\mathrm{t}}$, $\mathcal{D}'_{\mathrm{t}}$ be as in the previous section. As for the symbols $-^*$ and $-'$, the context will determine the interpretation (see Remark 2.7.5). To put a fine point to it, if $\mathscr{G}$ is in $\mathbf{coz}_{\Delta}(\mathscr{Y})$, then $\mathscr{G}^*=\mathscr{H}om(\mathscr{G},\mathscr{R})$, whereas if $\mathscr{G}\in\mathbf{coz}_{\Delta'}(\mathscr{X})$, then $\mathscr{G}^*=\mathscr{H}om(\mathscr{G},f^\sharp\mathscr{R})$. Similarly, $\mathscr{M}'$ is $\mathscr{H}om^\bullet(\mathscr{M},\mathscr{R})$ or $\mathscr{H}om^\bullet(\mathscr{M},f^\sharp\mathscr{R})$ depending on whether $\mathscr{M}$ is a coherent $\mathscr{O}_{\mathscr{Y}}$-module or a coherent $\mathscr{O}_{\mathscr{X}}$-module.

We denote by $\overline{Q}_{\mathscr{Y}}$ the localization functor

$$\overline{Q}_{\mathscr{Y}}\colon \mathbf{coz}_{\Delta}(\mathscr{Y})\to\mathbf{D}^*_{\mathrm{c}}(\mathscr{Y})\cap\mathbf{D}^+(\mathscr{Y}).$$

We would like to understand the effect of duality on Cousin complexes. In other words, we wish to study the functor

$$f^{(!)}\overline{Q}_{\mathscr{Y}}\colon \mathbf{coz}_{\Delta}(\mathscr{Y})\to\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})\cap\mathbf{D}^+(\mathscr{X}).$$

In order to describe the above functor more explicitly in terms of $\mathscr{R}$, we set

$$f^{(!)}_{[\mathscr{R}]}:=\mathcal{D}'_{\mathrm{t}}\circ\mathbf{L}f^*\circ Q_{\mathscr{Y}}(-)^*.$$

By (2.7.1) we have a canonical isomorphism $\mathcal{D}_{\mathrm{t}}\overline{Q}_{\mathscr{Y}}\xrightarrow{\sim}Q_{\mathscr{Y}}(-)^*$ of functors on $\mathbf{coz}_{\Delta}(\mathscr{Y})$. This induces a series of isomorphisms

$$f^{(!)}_{[\mathscr{R}]}\xrightarrow{\sim}f^{(!)}_{\mathscr{R}}\circ\overline{Q}_{\mathscr{Y}}\xrightarrow{\sim}f^{(!)}\circ\overline{Q}_{\mathscr{Y}}. \tag{4.3.1}$$

It is convenient—as we will see—to study $f^{(!)}\overline{Q}_{\mathscr{Y}}$ via $f^{(!)}_{[\mathscr{R}]}$. The behavior of $f^{(!)}_{[\mathscr{R}]}$ with respect to "change of residual complexes" obviously follows the behavior of $f^{(!)}_{\mathscr{R}}\overline{Q}_{\mathscr{Y}}$ with respect to such a change. In other words, if $\mathscr{R}'$ is another residual complex on $(\mathscr{Y},\Delta)$, we have an isomorphism of functors

$$\phi_{[\mathscr{R},\mathscr{R}']}\colon f^{(\sharp)}_{[\mathscr{R}']}\xrightarrow{\sim}f^{(\sharp)}_{[\mathscr{R}]}$$

which is compatible with $\phi_{\mathscr{R},\mathscr{R}'}$ and the first arrow in (4.3.1).

The behavior of $f^{(\sharp)}\overline{Q}_{\mathscr{Y}}$ is studied through a comparison map $\gamma^{(!)}_f\colon\overline{Q}_{\mathscr{X}}f^{(\sharp)}\to f^{(!)}\overline{Q}_{\mathscr{Y}}$ which is a more down to earth version of the comparison map in [S, p. 163, (4.1.4.1)] when we restrict our attention to $\mathbf{coz}_{\Delta}(\mathscr{Y})$ (instead of $\mathrm{Coz}_{\Delta}(\mathscr{Y})$). Here is how it is defined. Recall that if $\mathscr{M}\in\mathcal{A}_{\mathrm{c}}(\mathscr{X})$, then there is an obvious functorial map $\mathbf{L}f^*Q_{\mathscr{Y}}\mathscr{M}\to Q_{\mathscr{X}}f^*\mathscr{M}$. This induces a natural transformation

$$\gamma^*_f\colon\mathbf{L}f^*Q_{\mathscr{Y}}\to Q_{\mathscr{X}}f^*(-)^* \tag{4.3.2}$$

between functors on $\mathbf{coz}_\Delta(\mathscr{Y})$. Set

$$\gamma^{(!)}_{f,\mathscr{R}} := \mathcal{D}'_{\mathrm{t}}\gamma^*_f \colon \overline{Q}_{\mathscr{X}} f^{(\sharp)}_{\mathscr{R}} \to f^{(!)}_{[\mathscr{R}]}.$$

As can be easily checked from the definitions, this map behaves well with respect to change of residual complexes on $(\mathscr{Y},\Delta)$, i.e.

$$\phi_{[\mathscr{R},\mathscr{R}']}\gamma^{(!)}_{f,\mathscr{R}'} = \gamma^{(!)}_{f,\mathscr{R}}\overline{Q}_X\psi_{\mathscr{R},\mathscr{R}'}.$$

We therefore have a well-defined comparison map

$$\gamma^{(!)}_f \colon \overline{Q}_{\mathscr{X}} \circ f^{(\sharp)} \to f^{(!)} \circ \overline{Q}_{\mathscr{Y}}. \tag{4.3.3}$$

4.4. **Tor-independence.** The following definition does not need $\mathscr{X}$, $\mathscr{Y}$ or f to be in $\mathbb{F}^r$.

Definition 4.4.1. A pair $(f, \mathscr{M})$, with $f\colon \mathscr{X} \to \mathscr{Y}$ a map of formal schemes and $\mathscr{M}$ an object of $\mathcal{A}_{\mathrm{c}}(\mathscr{Y})$, is said to be a *tor-independent pair* if the following holds for every $x \in \mathscr{X}$ (with $y = f(x)$, $A = \mathscr{O}_{\mathscr{Y},y}$, $B = \mathscr{O}_{\mathscr{X},x}$ and $M = \mathscr{M}_y$):

$$\mathrm{Tor}^A_i(B,\, M) = 0 \qquad\qquad (i > o).$$

In other words, $(f, \mathscr{M})$ is tor-independent if and only if the natural map $\mathbf{L}f^*Q_{\mathscr{Y}} \to Q_{\mathscr{X}} f^*$ in $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$ is an isomorphism on $\mathscr{M}$:

$$\mathbf{L}f^*Q_{\mathscr{Y}}\mathscr{M} \xrightarrow{\ \sim\ } Q_{\mathscr{X}} f^*\mathscr{M}.$$

Remark 4.4.1.1. Note that $\mathscr{M}$ is a flat $\mathscr{O}_{\mathscr{Y}}$-module if and only if $(f, \mathscr{M})$ is tor-independent for every f. In fact, $\mathscr{M}$ is a flat $\mathscr{O}_{\mathscr{Y}}$-module if and only if $(f,\mathscr{M})$ is tor-independent for every closed immersion $f\colon \mathscr{X} \to \mathscr{Y}$.

Lemma 4.4.2. *Let $f\colon (\mathscr{X}, \Delta') \to (\mathscr{Y}, \Delta)$ be a map in $\mathbb{F}^r_{\mathrm{c}}$, $\mathscr{F}$ an object in $\mathbf{coz}_\Delta(\mathscr{Y})$, and $\mathscr{R}$ a residual complex on $(\mathscr{Y},\Delta)$. For $x \in \mathscr{X}$, let $y = f(x)$, $M = (\mathscr{F}^*)_y$ and A, B the local rings at x and y. Then for every integer i*

$$\mathrm{H}^i_x(f^{(!)}\mathscr{F}) \cong \mathrm{Hom}_B(\mathrm{Tor}^A_{i-\Delta'(x)}(B, M), f^\sharp\mathscr{R}(x)).$$

In particular, $f^{(!)}\mathscr{F} \in \mathbf{cm}(\mathscr{X};\Delta')$ if and only if $(f, \mathscr{F}^*)$ is a tor-independent pair (since the right side is the Matlis dual of the finitely generated B-module $\mathrm{Tor}^A_{i-\Delta'(x)}(B, M)$.

Proof. Since $f^\sharp\mathscr{R}$ is residual, whence injective, we have by [AJL1, p. 33, (5.2.1)]

$$\begin{aligned}\mathbf{R}\Gamma_x f^{(!)}\mathscr{F} &\cong \mathrm{Hom}^\bullet_B((\mathbf{L}f^*\mathscr{F}^*)_x,\, \Gamma_x\mathscr{R})\\ &\cong \mathrm{Hom}^\bullet_B(B\overset{\mathbf{L}}{\otimes}M,\, \mathscr{R}(x)[-\Delta'(x)])\\ &\cong \mathrm{Hom}^\bullet_B(B\overset{\mathbf{L}}{\otimes}M[\Delta'(x)],\, \mathscr{R}(x)).\end{aligned}$$

Since $\mathscr{R}(x)$ is an injective B-module, applying H^i to both sides, we get the result. □

Theorem 4.4.3. *Let $f\colon (\mathscr{X},\Delta') \to (\mathscr{Y},\Delta)$ and $\mathscr{F}$, $\mathscr{R}$ be as in the lemma above. The following are equivalent*

(i) *$f^{(!)}\mathscr{F}$ is Cohen-Macaulay with respect to Δ';*
(ii) *$(f, \mathscr{F}^*)$ is a tor-independent pair;*

(iii) *The map*

$$\gamma_f^*(\mathscr{F})\colon \mathbf{L}f^*Q_{\mathscr{Y}}\mathscr{F}^* \to Q_{\mathscr{X}}f^*\mathscr{F}^*$$

of (4.3.2) *is an isomorphism;*

(iv) *The map*

$$\gamma_f^{(!)}(\mathscr{F})\colon \overline{Q}_{\mathscr{X}}f^{(\sharp)}\mathscr{F} \to f^{(!)}\overline{Q}_{\mathscr{Y}}\mathscr{F}$$

of (4.3.3) *is an isomorphism.*

Proof. Evidently (i), (ii) and (iii) are equivalent. Since $\gamma_{f,\mathscr{R}}^{(!)}(\mathscr{F})$ is the "dual" of $\gamma_f^*(\mathscr{F})$ with respect to the residual complex $f^\sharp\mathscr{R}$, clearly (iv) is equivalent to (iii). □

Theorem 4.4.3 gives us a way of reproving (and allows for a better understanding of) [S, p. 191, Thm. 9.3.12] (cf. [S, p. 182, Thm. 7.2.2]). Moreover, coupled with [S, p. 191, Thm. 9.3.13] it allows for subtle twist on that theorem on Gorenstein complexes. We should point out that there is a typographical error in *loc.cit.*—the hypothesis on $\mathcal{F}$ should be $\mathcal{F} \in \mathbf{coz}_\Delta(\mathscr{Y})$ and not $\mathcal{F} \in \mathrm{Coz}_\Delta(\mathscr{Y})$.

Theorem 4.4.4. [S, 9.3.12 and 7.2.2] *Let f and $\mathscr{R}$ be as above. Then the following are equivalent*

(i) *f is flat;*

(ii) *$f^{(!)}\mathscr{F}$ is Cohen-Macaulay with respect to Δ' for every $\mathscr{F} \in \mathbf{cm}(\mathscr{Y};\Delta)$;*

(iii) *The map of functors*

$$\gamma_f^{(!)}\colon \overline{Q}_{\mathscr{X}}f^{(\sharp)} \to f^{(!)}\overline{Q}_{\mathscr{Y}}$$

is an isomorphism.

Proof. This follows immediately from Theorem 4.4.3 and the fact that f is flat if and only if $\gamma_f^*\colon \mathbf{L}f^*Q_{\mathscr{Y}}\mathscr{F}^* \to Q_{\mathscr{X}}f^*\mathscr{F}^*$ is an isomorphism for every $\mathscr{F} \in \mathbf{coz}_\Delta(\mathscr{Y})$. We point out that the essential image of $\mathbf{coz}_\Delta(\mathscr{Y})$ under $-'$ is $\mathcal{A}_{\mathrm{c}}(\mathscr{Y})$ according to Proposition 2.7.4. □

We now move to examining another statement in [S]. In [S, p. 178, Thm. 6.3.1] it is shown that the Cousin of the map $\gamma_f^!$ is an isomorphism. It is much simpler to prove the analogous statement for $\gamma_f^{(!)}$. It is worth pointing out that this analogous statement gains strength only when content is poured into it by showing that $\gamma_f^{(!)}$ is "isomorphic" to $\gamma_f^!$, i.e., by showing Diagram (5.3.2.1) commutes. Note this involves showing that $f^{(\sharp)}$ is isomorphic to $f^\sharp$ (when restricted to Cousin complexes in $\mathbf{D}_{\mathrm{c}}^*$) and that $f^!$ is isomorphic to $f^{(!)}$ (when restricted to $\mathbf{D}_{\mathrm{c}}$). The latter is proven in [S] using the fact that $\gamma_f^!$ is an isomorphism for residual complexes (in itself needing a careful examination of $\gamma_f^!$) and the former needs Theorem 5.3.3. In other words, the "simpler" proof gains content only when many more complicated proofs are brought in to set the context. Here then is the statement analogous to [S, p. 178, Thm. 6.3.1]. Let $f\colon (\mathscr{X},\Delta') \to (\mathscr{Y},\Delta)$ be a map in $\mathbb{F}_{\mathrm{c}}^r$, and set

$$f^{(E)} := E_{\Delta'}f^{(!)}\overline{Q}_{\mathscr{Y}}\colon \mathbf{coz}_\Delta(\mathscr{Y}) \to \mathbf{coz}_{\Delta'}(\mathscr{X})$$

and $\gamma_f^{(E)}$ to be the composite

$$f^{(\sharp)} \xrightarrow{\ \sim\ } E_{\Delta'}(f^{(\sharp)}) \xrightarrow{E(\gamma_f^{(!)})} f^{(E)}.$$

We then have

Proposition 4.4.5. *The functorial map*

$$\gamma_f^{(E)}\colon f^{(\sharp)} \to f^{(E)}$$

is an isomorphism.

Proof. Fix a residual complex $\mathscr{R}$ on $(\mathscr{Y},\Delta)$. It is enough to show that the functorial map $E(\gamma_{\mathscr{R}}^{(!)})\colon E(f_{\mathscr{R}}^{(\sharp)}) \to E(f_{\mathscr{R}}^{(!)})$ is an isomorphism or what amounts to the same thing, that

$$\mathrm{H}_x^{\Delta'(x)}(\gamma_{\mathscr{R}}^{(!)})\colon \mathrm{H}_x^{\Delta'(x)}(f_{\mathscr{R}}^{(\sharp)}) \to \mathrm{H}_x^{\Delta'(x)}(f_{\mathscr{R}}^{(!)})$$

is an isomorphism for every $x \in \mathscr{X}$. Fixing such an x, we see—as in the proof of Lemma 4.4.2—that after taking Matlis duals this amounts to showing that for $\mathscr{F} \in \mathbf{coz}_\Delta(\mathscr{Y})$, the natural map

$$H^0((\gamma_f^*))_x\colon H^0(\mathbf{L}f^*\mathscr{F}^*)_x \to H^0(f^*\mathscr{F}^*)_x$$

is an isomorphism, which it clearly is. $\square$

5. The pseudofunctor $-^{(\sharp)}$ vs. the pseudofunctor $-^{\sharp}$

In this section we show that $f^{(\sharp)}\mathscr{F}$ is naturally isomorphic to $f^{\sharp}\mathscr{F}$ when $\mathscr{F} \in \mathbf{coz}_\Delta(\mathscr{Y})$ and $f\colon (\mathscr{X},\Delta') \to (\mathscr{Y},\Delta)$ is a map in $\mathbb{F}_{\mathrm{c}}^r$. But first we wish to understand the behavior of $(fg)^{(\sharp)}$ for a composite of two maps f and g with respect to $f^{(\sharp)}$ and $g^{(\sharp)}$.

5.1. **Variance properties.** We assume familiarity with the notion of a *contravariant pseudofunctor* defined for example in [LNS, p. 45]. Indeed the main focus of [LNS] is to construct $f^{\sharp}$ for suitable maps f in such a way that the assignments $(\mathscr{Y},\Delta) \mapsto \mathrm{Coz}_\Delta(\mathscr{Y})$ and $f \mapsto f^{\sharp}$ define a pseudofunctor $-^{\sharp}$. It turns out that the assignments $(\mathscr{Y},\Delta) \mapsto \mathbf{coz}_\Delta(\mathscr{Y})$, $(\mathscr{Y},\Delta) \in \mathbb{F}_{\mathrm{c}}^r$, and $f \mapsto f^{(\sharp)}$, f a map in $\mathbb{F}_{\mathrm{c}}^r$, are pseudofunctorial. To see this, let

$$(\mathscr{W},\Delta'') \xrightarrow{g} (\mathscr{X},\Delta') \xrightarrow{f} (\mathscr{Y},\Delta)$$

be a pair of maps in $\mathbb{F}_{\mathrm{c}}^r$. Let $\mathscr{R}$ be a residual complex on $(\mathscr{Y},\Delta)$ and $\mathscr{S} := f^{\sharp}\mathscr{R}$. The pseudofunctor $-^{\sharp}$ gives an isomorphism

$$C_{g,f}^{\sharp}(\mathscr{R})\colon g^{\sharp}f^{\sharp}\mathscr{R} \xrightarrow{\ \sim\ } (fg)^{\sharp}\mathscr{R}.$$

This together with the isomorphisms $f^*\mathscr{F}^* \xrightarrow{\ \sim\ } (f^*\mathscr{F}^*)'^* = (f^{(\sharp)}\mathscr{F})^*$ (cf. Remark 2.7.5) gives an isomorphism

$$C_{g,f,\mathscr{R}}^{(\sharp)}\colon g_{\mathscr{S}}^{(\sharp)}f_{\mathscr{R}}^{(\sharp)} \xrightarrow{\ \sim\ } (fg)_{\mathscr{R}}^{(\sharp)}.$$

The process is completely analogous to the one described [C, p. 136, (3.3.15)] and [S, p. 188, (9.2.3)] for $-^{(!)}$. The isomorphism $C_{g,f,\mathscr{R}}^{(\sharp)}$ behaves well with respect to change of residual complexes, giving an isomorphism

$$C_{g,f}^{(\sharp)}\colon g^{(\sharp)}f^{(\sharp)} \xrightarrow{\ \sim\ } (fg)^{(\sharp)}.$$

Using the pseudofunctoriality of $-^{\sharp}$ it is easy to see that the above identification is "associative", and hence defines a pseudofunctor $-^{(\sharp)}$ on $\mathbb{F}_{\mathrm{c}}^r$ with $(\mathscr{Y},\Delta)^{(\sharp)} = \mathbf{coz}_\Delta(\mathscr{Y})$ for $(\mathscr{Y},\Delta) \in \mathbb{F}_{\mathrm{c}}^r$. Since, as we briefly noted, the process is identical to the process of constructing the pseudofunctor $-^{(!)}$, with f^*, g^* and $(fg)^*$ replacing $\mathbf{L}f^*, \mathbf{L}g^*$ and $\mathbf{L}(fg)^*$ in the construction in [S, p. 188, (9.2.3)], we have the following proposition (cf. [S, p. 163, Thm. 4.1.4(d)]):

Proposition 5.1.1. *With f, g as above, the following diagram commutes:*

$$\begin{tikzcd}
\overline{Q}_{\mathscr{W}} g^{(\sharp)} f^{(\sharp)} \arrow[r, "C^{(\sharp)}_{g,f}"] \arrow[d, "\gamma^{(!)}_g(f^{(\sharp)})"'] & \overline{Q}_{\mathscr{W}} (fg)^{(\sharp)} \arrow[dd, "\gamma^{(!)}_{fg}"] \\
g^{(!)}\overline{Q}_{\mathscr{X}} f^{(\sharp)} \arrow[d, "g^{(!)}(\gamma^{(!)}_f)"'] & \\
g^{(!)} f^{(!)}\overline{Q}_{\mathscr{Y}} \arrow[r, "C^{(!)}_{g,f}"] & (fg)^{(!)}\overline{Q}_{\mathscr{Y}}
\end{tikzcd}$$

where the map $C^{(!)}_{g,f}$ is the map in [S, p. 188, (9.2.3)].

5.2. **$-^{(\sharp)}$ vs. $-^{\sharp}$.** For $f\colon (\mathscr{X}, \Delta') \to (\mathscr{Y}, \Delta)$ and $\mathscr{F} \in \mathbf{coz}_{\Delta}(\mathscr{Y})$ define a map

$$\zeta = \zeta_f(\mathscr{F})\colon f^{\sharp}\mathscr{F} \to f^{(\sharp)}\mathscr{F}$$

as follows. Pick a residual complex $\mathscr{R}$ on $(\mathscr{Y}, \Delta)$. Functoriality of $f^{\sharp}$ gives a map $\Gamma(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}})$-modules $\mathrm{Hom}(\mathscr{F}, \mathscr{R}) \to \mathrm{Hom}(f^{\sharp}\mathscr{F}, f^{\sharp}\mathscr{R})$ which is well behaved with respect to Zariski localizations of $\mathscr{Y}$. In other words we have a map of $\mathscr{O}_{\mathscr{Y}}$-modules

$$\mathscr{F}^* = \mathscr{H}om(\mathscr{F}, \mathscr{R}) \to f_*\mathscr{H}om(f^{\sharp}\mathscr{F}, f^{\sharp}\mathscr{R}) = f_*((f^{\sharp}\mathscr{F})^*)$$

inducing a map of coherent $\mathscr{O}_{\mathscr{X}}$-modules

$$\xi = \xi_f(\mathscr{F})\colon f^*(\mathscr{F}^*) \to (f^{\sharp}\mathscr{F})^*.$$

The natural isomorphism $f^{\sharp}\mathscr{F} \xrightarrow{\ \sim\ } (f^{\sharp}\mathscr{F})^{*'}$ of Proposition 2.7.4 followed by ξ' gives us a map

$$\zeta_{\mathscr{R}}\colon f^{\sharp}\mathscr{F} \to f^{(\sharp)}_{\mathscr{R}}\mathscr{F}$$

which one checks (from the definitions) is independent of $\mathscr{R}$, i.e.

$$\psi_{[\mathscr{R},\mathscr{R}']}(\mathscr{F}) \circ \zeta_{\mathscr{R}'} = \zeta_{\mathscr{R}}.$$

We therefore get a well defined map of functors

$$\zeta_f\colon f^{\sharp}|_{\mathbf{coz}\,(\mathscr{Y})} \to f^{(\sharp)} \tag{5.2.1}$$

If $g\colon (W, \Delta'') \to (\mathscr{X}, \Delta')$ is a second map, it is easy to check from the definitions that the diagram

$$\begin{tikzcd}
g^{\sharp}f^{\sharp}\mathscr{F} \arrow[r, "g^{\sharp}\zeta_f"] \arrow[d, "C^{\sharp}_{g,f}"', "\wr"] & g^{\sharp}f^{(\sharp)}\mathscr{F} \arrow[r, "\zeta_g"] & g^{(\sharp)}f^{(\sharp)}\mathscr{F} \arrow[d, "\wr"', "C^{(\sharp)}_{g,f}"] \\
(fg)^{\sharp}\mathscr{F} \arrow[rr, "\zeta_{fg}"'] & & (fg)^{(\sharp)}\mathscr{F}
\end{tikzcd} \tag{5.2.2}$$

commutes for every $\mathscr{F} \in \mathbf{coz}_{\Delta}(\mathscr{Y})$.

5.3. **Traces.** Let $f\colon (\mathscr{X}, \Delta') \to (\mathscr{Y}, \Delta)$ be a pseudo-proper map in $\mathbb{F}_{\mathrm{c}}^{r}$. According to [S, p. 146, (2.2.4)] and [S, p. 156, Thm. 2.4.2(b)], for every $\mathscr{F} \in \mathrm{Coz}_{\Delta}(\mathscr{Y})$ we have a trace map

$$\mathrm{Tr}_f(\mathscr{F})\colon f_*f^{\sharp}\mathscr{F} \to \mathscr{F}.$$

If $\mathscr{F} \in \mathbf{coz}_{\Delta}(\mathscr{Y}) \subset \mathrm{Coz}_{\Delta}(\mathscr{Y})$, then we define, as a counterpart to Tr_f,

$$\mathrm{Tr}_f^{(\sharp)}(\mathscr{F})\colon f_*f^{(\sharp)}\mathscr{F} \to \mathscr{F} \tag{5.3.1}$$

as follows. First pick a residual complex $\mathscr{R}$ on $(\mathscr{Y}, \Delta)$ and define $\mathrm{Tr}_{f,\mathscr{R}}^{(\sharp)}(\mathscr{F})$ as the map which makes the following diagram commute (see also [S, p. 189, (9.3.5)]):

$$\begin{tikzcd}
f_*f_{\mathscr{R}}^{(\sharp)}\mathscr{F} \arrow[r, equal] \arrow[d, "\mathrm{Tr}_{f,\mathscr{R}}^{(\sharp)}(\mathscr{F})"'] & f_*\mathscr{H}om^{\bullet}(f^*\mathscr{F}^*, f^{\sharp}\mathscr{R}) \arrow[r, "\sim"] & \mathscr{H}om^{\bullet}(\mathscr{F}^*, f_*f^{\sharp}\mathscr{R}) \arrow[d, "\mathrm{Tr}_f(\mathscr{R})"] \\
\mathscr{F} \arrow[r, "\sim"] & \mathscr{F}^{*\prime} \arrow[r, equal] & \mathscr{H}om^{\bullet}(\mathscr{F}^*, \mathscr{R})
\end{tikzcd}$$

As usual, one checks that this definition is independent of $\mathscr{R}$, i.e. we have a relation $\mathrm{Tr}_{f,\mathscr{R}}^{(\sharp)} f_*\psi_{[\mathscr{R},\mathscr{R}']} = \mathrm{Tr}_{f,\mathscr{R}'}^{(\sharp)}$. This gives (5.3.1).

We had, just before the above definition, fleetingly drawn the reader's attention to the trace map in [S, p. 189, (9.3.5)]

$$\tau_f^r\colon \mathbf{R}f_*f^{(!)} \to \mathbf{1}.$$

The point is that the definition of $\mathrm{Tr}_f^{(\sharp)}$ is almost identical to the definition of τ_f^r, provided we replace f^* by $\mathbf{L}f^*$, and this gives part (iii) of the Proposition 5.3.2 below. Part (i) is immediate from the analogous [S, p. 156, Thm. 2.4.2(b)] and part (ii) is immediate from the definition of $\mathrm{Tr}_f^{(\sharp)}$.

Proposition 5.3.2. *Let $f\colon (\mathscr{X}, \Delta') \to (\mathscr{Y}, \Delta)$ be a pseudo-proper map in $\mathbb{F}_{\mathrm{c}}^{r}$ and $\mathscr{F} \in \mathbf{coz}_{\Delta}(\mathscr{Y})$.*

(i) *If $g\colon (W, \Delta'') \to (\mathscr{X}, \Delta')$ is a second pseudo-proper map then the diagram*

$$\begin{tikzcd}
(fg)_*g^{(\sharp)}f^{(\sharp)}\mathscr{F} \arrow[r, "\sim", "C_{g,f}^{(\sharp)}"'] \arrow[d, equal] & (fg)_*(fg)^{(\sharp)}\mathscr{F} \arrow[dd, "\mathrm{Tr}_{fg}^{(\sharp)}"] \\
f_*g_*g^{(\sharp)}f^{(\sharp)}\mathscr{F} \arrow[d, "f_*\mathrm{Tr}_g^{(\sharp)}"'] & \\
f_*f^{(\sharp)}\mathscr{F} \arrow[r, "\mathrm{Tr}_f^{(\sharp)}"'] & \mathscr{F}
\end{tikzcd}$$

commutes (see [S, p. 156, Thm. 2.4.2(b)]*).*

(ii) *The diagram*

$$\begin{tikzcd}
f_*f^{\sharp}\mathscr{F} \arrow[r, "f_*\zeta_f"] \arrow[dr, "\mathrm{Tr}_f"'] & f_*f^{(\sharp)}\mathscr{F} \arrow[d, "\mathrm{Tr}_f^{(\sharp)}"] \\
& \mathscr{F}
\end{tikzcd}$$

commutes.

(iii) *The diagram (in which we suppress localization functors like $\overline{Q}_{\mathscr{Y}}$ to avoid clutter)*

$$\begin{tikzcd} f_*f^{(\sharp)}\mathscr{F} \arrow[r, "\sim"] \arrow[d, "\mathrm{Tr}_f^{(\sharp)}"'] & \mathbf{R}f_*f^{(\sharp)}\mathscr{F} \arrow[d, "\mathbf{R}f_*\gamma_f^{(!)}"] \\ \mathscr{F} & \mathbf{R}f_*f^{(!)}\mathscr{F} \arrow[l, "\tau_f^r"] \end{tikzcd}$$

commutes in $\mathbf{D}_{\mathrm{c}}^*(\mathscr{Y})\cap\mathbf{D}^+(\mathscr{Y})$.

If f and $\mathscr{F}$ are as in the Proposition and

$$\Phi_f(\mathscr{F})\colon f^{(!)}\mathscr{F} \xrightarrow{\ \sim\ } f^!\mathscr{F}$$

is the isomorphism in [S, p. 190, Thm. 9.3.10] then by Proposition 5.3.2(ii) and (iii) and the universal properties of $(f^!,\tau_f)$ and $(f^{(!)},\tau_f^r)$ the following diagram

(5.3.2.1)
$$\begin{tikzcd} f^\sharp\mathscr{F} \arrow[r, "\zeta_f"] \arrow[d, "\gamma_f^!"'] & f^{(\sharp)}\mathscr{F} \arrow[d, "\gamma_f^{(!)}"] \\ f^!\mathscr{F} & f^{(!)}\mathscr{F} \arrow[l, "\sim", "\Phi_f"'] \end{tikzcd}$$

commutes in $\mathbf{D}_{\mathrm{c}}^*(\mathscr{X})\cap\mathbf{D}^+(\mathscr{X})$, where $\gamma_f^!$ is the map in [S, p. 163, (4.1.4.1)].

Here is how we compare $-^\sharp$ and $-^{(\sharp)}$. Recall that a compactifiable map is a map that can be written as an open immersion followed by a pseudo-proper map.

Theorem 5.3.3. *Let $f\colon(\mathscr{X},\Delta')\to(\mathscr{Y},\Delta)$ be a map in $\mathbb{F}_{\mathrm{c}}^r$.*

(i) *The map $\zeta_f\colon f^\sharp|_{\mathbf{coz}_\Delta(\mathscr{Y})}\to f^{(\sharp)}$ is an isomorphism of functors.*
(ii) *Diagram* (5.3.2.1) *continues to commute under the weaker hypothesis that f is a composite of compactifiable maps.*

Proof. We first prove (ii). By (5.2.2), Proposition 5.1.1, [S, p. 163, Thm. 4.1.4(d)] and [*Ibid*, p. 190, (9.3.10.1)] the maps ζ_f, $\gamma_f^{(!)}$, $\gamma_f^!$ and Φ_f behave well with respect to composition of maps. Therefore it is enough to prove that (5.3.2.1) commutes when f is pseudo-proper and when f is an open immersion. We have already argued that the diagram commutes when f is pseudo-proper. If f is an open immersion, all vertices in the diagram can be identified with $f^*\mathscr{F}$ and all arrows with the identity map, and hence we are done.

Part (i) is equivalent to showing that $E_{\Delta'}(\zeta_f)$ is an isomorphism. Moreover the question is local on $\mathscr{X}$, and therefore we may assume that f is a composite of compactifiable maps. We have proven that in this case (5.3.2.1) commutes. Applying $E_{\Delta'}$ to this diagram, and using the fact that $E_{\Delta'}(\gamma_f^!)$ and $E_{\Delta'}(\gamma_f^{(!)})$ are isomorphisms by [S, p. 178 Thm. 6.3.1] and Proposition 4.4.5, we are done. $\square$

One consequence of Theorem 5.3.3 is that every $\mathbf{C}(\mathscr{X})$-map $f^\sharp\mathscr{F}\to f^\sharp\mathscr{R}$ is induced by a $\mathbf{C}(\mathscr{Y})$-map $\mathscr{F}\to\mathscr{R}$. More precisely, we have:

Corollary 5.3.4. *Let $\mathscr{F}$ be an object in $\mathbf{coz}_\Delta(\mathscr{Y})$ and $\mathscr{R}$ a residual complex on $(\mathscr{Y},\Delta)$. The natural map*

$$\xi_f\colon f^*\mathscr{H}om_{\mathscr{Y}}(\mathscr{F},\mathscr{R})\to\mathscr{H}om_{\mathscr{X}}(f^\sharp\mathscr{F},f^\sharp\mathscr{R})$$

is an isomorphism.

Proof. By construction of ζ_f, ξ_f is the dual (with respect to $\mathscr{R}$) of ζ_f, which we have shown is an isomorphism. □

6. Gorenstein complexes

In this section we drop the assumption that our formal schemes contain c-dualizing complexes. In Subsection 6.1 we work with general noetherian schemes. Starting from Subsection 6.2 we restrict ourselves to schemes and maps in $\mathbb{F}$ (which, recall from Subsection 2.1 is the category whose objects are noetherian formal schemes which are universally catenary and admit codimension functions).

6.1. **The homology localization functor and the derived torsion functor.** The purpose of this subsection is to recall (and draw out) the various ways in which the "homology localization functor" of [AJL2] and [AJL1] interacts with the derived torsion functor, as summarized in [AJL2, pp. 69–70, Remarks 6.3.1(1)]. In greater detail, let $\mathscr{X}$ be a formal scheme (not necessarily in $\mathbb{F}$, but as always, noetherian). The homology localization functor $\mathbf{\Lambda}_{\mathscr{X}}\colon \mathbf{D}(\mathscr{X}) \to \mathbf{D}(\mathscr{X})$ is defined as

$$\mathbf{\Lambda}_{\mathscr{X}} := \mathbf{R}\mathscr{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{O}_{\mathscr{X}},\, -)$$

where $\varGamma'_{\mathscr{X}}$ is the torsion functor defined in Subsection 2.2. Note that the source and the target of $\mathbf{R}\varGamma'_{\mathscr{X}}$ is $\mathbf{D}(\mathscr{X})$. For typographical convenience we write $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathscr{X}}$ and $\boldsymbol{\varGamma} = \mathbf{R}\varGamma'_{\mathscr{X}}$. By [AJL2, p. 54, 5.2.10.1] we have $\mathbf{\Lambda}$ is right adjoint to $\boldsymbol{\varGamma}$. More precisely, we have a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(\mathscr{E},\, \mathbf{\Lambda}\mathscr{F}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(\boldsymbol{\varGamma}\mathscr{E},\, \mathscr{F}) \qquad (\mathscr{E}, \mathscr{F} \in \mathbf{D}(\mathscr{X})).$$

We are more interested in the sheafified version of the above, obtained by noting that $\boldsymbol{\varGamma}$ and $\mathbf{\Lambda}$ are compatible with Zariski localization as is the above bifunctorial map (see also the comment preceding [AJL2, p. 24, (2.5.0.1)]), namely:

$$\mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{E},\, \mathbf{\Lambda}\mathscr{F}) \xrightarrow{\ \sim\ } \mathbf{R}\mathscr{H}om^{\bullet}(\boldsymbol{\varGamma}\mathscr{E},\, \mathscr{F}) \qquad (\mathscr{E}, \mathscr{F} \in \mathbf{D}(\mathscr{X})). \tag{6.1.1}$$

The above also needs the fact that the adjointness between $\boldsymbol{\varGamma}$ and $\mathbf{\Lambda}$ is δ-functorial, i.e., it behaves well with translations. There are three results (other than (6.1.1))) that are important for us in this paper:

(a) According to [AJL2, p. 68, Prop. 6.2.1], a form of the Greenlees-May duality, we have

$$\mathbf{\Lambda}\mathscr{F} \xrightarrow{\ \sim\ } \mathscr{F} \qquad (\mathscr{F} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})). \tag{6.1.2}$$

(b) For $\mathscr{E} \in \mathbf{D}(\mathscr{X})$ and $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$ we have

$$\mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{E},\, \mathscr{F}) \xrightarrow{\ \sim\ } \mathbf{R}\mathscr{H}om^{\bullet}(\boldsymbol{\varGamma}\mathscr{E},\, \boldsymbol{\varGamma}\mathscr{F}). \tag{6.1.3}$$

(c) The functors $\mathbf{\Lambda}$ and $\boldsymbol{\varGamma}$ induce quasi-inverse equivalences between the categories $\mathbf{D}^{*}_{\mathrm{c}}(\mathscr{X})$ and $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$. In other words we have (with $\mathbf{D}\hat{\ }(\mathscr{X})$ the essential image of $\mathbf{D}(\mathscr{X})$ under $\mathbf{\Lambda}$):

$$\mathscr{E} \in \mathbf{D}^{*}_{\mathrm{c}}(\mathscr{X}) \iff \mathbf{\Lambda}\mathscr{E} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})\,\mathrm{and}\,\mathscr{E} \in \mathbf{D}_{\mathrm{t}}(\mathscr{X}),$$
$$\mathscr{F} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X}) \iff \boldsymbol{\varGamma}\mathscr{F} \in \mathbf{D}^{*}_{\mathrm{c}}(\mathscr{X})\,\mathrm{and}\,\mathscr{F} \in \mathbf{D}\hat{\ }(\mathscr{X})$$

and

$$\mathscr{E} \xrightarrow{\ \sim\ } \boldsymbol{\varGamma}\mathbf{\Lambda}\mathscr{E} \qquad (\mathscr{E} \in \mathbf{D}^{*}_{\mathrm{c}}(\mathscr{X}))$$
$$\mathscr{F} \xrightarrow{\ \sim\ } \mathbf{\Lambda}\boldsymbol{\varGamma}\mathscr{F} \qquad (\mathscr{F} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X}))$$

Statement (b) is proven as follows. By [AJL2, pp. 69–70, Rmks 6.3.1 (1)(c)], we have

$$\mathbf{\Lambda}\boldsymbol{\varGamma} \xrightarrow{\ \sim\ } \mathbf{\Lambda}. \tag{6.1.4}$$

The relations (6.1.2), (6.1.4) and (6.1.1) then give—for $\mathscr{E} \in \mathbf{D}(\mathscr{X})$ and $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$—the sequence of isomorphisms

$$\begin{aligned}\mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{E},\, \mathscr{F}) &\xrightarrow{\ \sim\ } \mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{E},\, \mathbf{\Lambda}\mathscr{F})\\ &\xrightarrow{\ \sim\ } \mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{E},\, \mathbf{\Lambda}\boldsymbol{\varGamma}\mathscr{F})\\ &\xrightarrow{\ \sim\ } \mathbf{R}\mathscr{H}om^{\bullet}(\boldsymbol{\varGamma}\mathscr{E},\, \boldsymbol{\varGamma}\mathscr{F})\end{aligned}$$

thus establishing (6.1.3). We should point out that the map underlying the isomorphism (6.1.3) is the one induced by the functor $\boldsymbol{\varGamma}$. This can be seen by unravelling the definitions of the maps and isomorphisms in [AJL2, pp. 69–70, Rmks 6.3.1 (1)].

Statement (c) follows from the fact that $\mathbf{\Lambda}$ and $\boldsymbol{\varGamma}$ induce quasi-inverse equivalences between $\mathbf{D}_{\mathrm{t}}(\mathscr{X})$ and $\mathbf{D}\hat{\ }(\mathscr{X})$ (see the last line of [AJL2, Remarks 6.3.1 (1)]).

Remark 6.1.5. From [AJL2, Remarks 6.3.1 (1)] it is apparent that $\boldsymbol{\varGamma}$ and $\mathbf{\Lambda}$ share a symmetric relationship (e.g. $\boldsymbol{\varGamma}^2 \cong \boldsymbol{\varGamma}$, $\mathbf{\Lambda}^2 \cong \mathbf{\Lambda}$, $\boldsymbol{\varGamma}\mathbf{\Lambda} \cong \boldsymbol{\varGamma}$ and $\mathbf{\Lambda} \cong \mathbf{\Lambda}\boldsymbol{\varGamma}$). It is not hard to see that one can obtain an isomorphism "dual" to (6.1.3) given by

$$\mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{E},\, \mathscr{F}) \xrightarrow{\ \sim\ } \mathbf{R}\mathscr{H}om^{\bullet}(\mathbf{\Lambda}\mathscr{E},\, \mathbf{\Lambda}\mathscr{F}) \qquad (\mathscr{E} \in \mathbf{D}^{*}_{\mathrm{c}}(\mathscr{X}), \mathscr{F} \in \mathbf{D}(\mathscr{X})).$$

Again, the map underlying this isomorphism is the obvious one induced by the functor $\mathbf{\Lambda}$.

6.2. **c-Gorenstein and t-Gorenstein complexes.** In this subsection we assume that our schemes and maps of schemes are in $\mathbb{F}$.

Definition 6.2.1. Let $(\mathscr{X}, \Delta) \in \mathbb{F}_c$. A complex $\mathscr{F} \in \mathbf{D}(\mathscr{X})$ is said to be *t-Gorenstein* with respect to Δ if

(a) $\mathscr{F} \in \mathbf{D}_{\mathrm{qct}}(\mathscr{X})$;
(b) $\mathscr{F}$ is Cohen-Macaulay with respect to Δ; and
(c) the Cousin complex of $\mathscr{F}$ with respect to Δ, $E_{\Delta}\mathscr{F}$, consists of injective objects in $\mathcal{A}_{\mathrm{qct}}(\mathscr{X})$.

A complex $\mathscr{G} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$ is said to be *c-Gorenstein* with respect to Δ if $\boldsymbol{\varGamma}\mathscr{G}$ is t-Gorenstein.

Remarks 6.2.2.

(1) For ordinary schemes of finite Krull dimension, the notions of c-Gorenstein and t-Gorenstein coincide, and in this situation we call such complexes simply Gorenstein. A t-dualizing complex is t-Gorenstein and a c-dualizing complex is c-Gorenstein (cf. Definition 2.3.1 and Example 2.3.2).
(2) What is called simply *Gorenstein* in [S, p. 179] is the same as what we have called a t-Gorenstein complex in this paper.
(3) If $\mathscr{X}$ has finite Krull dimension, a t-Gorenstein complex, by this definition, is necessarily in $\mathbf{D}^{b}_{\mathrm{qct}}(\mathscr{X})$.
(4) We have the following two relations:
Let $\mathscr{F} \in \mathbf{D}^{*}_{\mathrm{c}}(\mathscr{X})$. Then $\mathscr{F}$ is t-Gorenstein $\iff$ $\mathbf{\Lambda}\mathscr{F}$ is c-Gorenstein.
Let $\mathscr{G} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$. Then $\mathscr{G}$ is c-Gorenstein $\iff$ $\boldsymbol{\varGamma}\mathscr{G}$ is t-Gorenstein.

Our immediate aim is to prove that if $\mathscr{G}_1$ and $\mathscr{G}_2$ are c-Gorenstein and $\mathscr{X}$ is of finite Krull dimension, then the complex $\mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{G}_1, \mathscr{G}_2)$ is $\mathbf{D}(\mathscr{X})$ isomorphic to a locally free $\mathscr{O}_{\mathscr{X}}$-module of finite rank, i.e. Proposition 6.2.5. We need a

preliminary discussion before we can do this. To that end, let $\mathscr{X}$ be a (noetherian) formal scheme. The torsion functor $\varGamma'_{\mathscr{X}}$ defined in Subsection 2.2 is a special case of the functor $\varGamma_{\mathscr{I}}\colon \mathcal{A}(\mathscr{X}) \to \mathcal{A}(\mathscr{X})$ for any $\mathscr{O}_{\mathscr{X}}$ ideal $\mathscr{I}$ (not necessarily an ideal of definition of the formal scheme $\mathscr{X}$). This is defined, as in [AJL2, p. 6, 1.2.1], by the formula

$$\varGamma_{\mathscr{I}} := \varinjlim_{n} \mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(\mathscr{O}_{\mathscr{X}}/\mathscr{I}^n,\, -).$$

Note that if $\mathscr{I}$ is an ideal of definition for $\mathscr{X}$ then $\varGamma_{\mathscr{I}} = \varGamma'_{\mathscr{X}}$.

For $x \in \mathscr{X}$, let $k(x)$ denote the residue field of the local ring $A := \mathscr{O}_{\mathscr{X},x}$, and let $E(x)$ denote the injective hull (as an A-module) of $k(x)$. It is well known that $E(x)$ is also an $\widehat{A}$-module, $\widehat{A}$ being the completion of A with respect to the maximal ideal $\mathfrak{m}_A$ of A. Moreover $E(x)$ is also the injective hull of $k(x)$ viewed as an $\widehat{A}$-module. If x is a *closed point* of $\mathscr{X}$ we denote the ideal sheaf of $\{x\}$ by $\mathfrak{m}_x$. Let

$$\kappa\colon \widehat{\mathscr{X}} \to \mathscr{X}$$

be the completion of $\mathscr{X}$ along $\{x\}$, then, denoting the unique point of $\widehat{\mathscr{X}}$ by $\widehat{x}$, we have $\kappa^* i_x E(x) = i_{\hat{x}} E(x)$. We denote the common $\mathscr{O}_{\widehat{\mathscr{X}}}$-module $\mathscr{E}(x)$. For $\mathscr{G} \in \mathbf{D}(\mathscr{X})$ we have, by [AJL2, p.50, Proposition 5.2.4], canonical functorial isomorphisms (each map deducible from the other via the adjoint pair (κ^*, κ_*))

$$\begin{aligned} \kappa^*\mathbf{R}\varGamma_{\mathfrak{m}_x}\mathscr{G} &\xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\widehat{\mathscr{X}}}\kappa^*\mathscr{G} \\ \mathbf{R}\varGamma_{\mathfrak{m}_x}\mathscr{G} &\xrightarrow{\ \sim\ } \kappa_*\mathbf{R}\varGamma'_{\widehat{\mathscr{X}}}\kappa^*\mathscr{G}. \end{aligned} \tag{6.2.3}$$

We point out that κ being a flat map of locally ringed spaces, $\kappa^* = \mathbf{L}\kappa^*$. Suppose now that Δ is a codimension function on $\mathscr{X}$ such that $(\mathscr{X}, \Delta) \in \mathbb{F}_c$, and suppose further that $\mathscr{G}$ is c-Gorenstein with respect to Δ. The complex $\mathscr{G}$ being c-Gorenstein, $\mathbf{R}\varGamma_{\mathfrak{m}_x}\mathscr{G}$ is isomorphic to a direct sum of copies of the complex $i_x E(x)[-\Delta(x)]$, whence from (6.2.3).

$$\mathbf{R}\varGamma'_{\widehat{\mathscr{X}}}\kappa^*\mathscr{G} \cong \oplus_{i\in I}\mathscr{E}(x)[-\Delta(x)]. \tag{6.2.4}$$

where the index i varies over a finite index set I. To see the finiteness of I, note that $\mathscr{E}(x)[-\Delta(x)]$ is t-dualizing on $\widehat{\mathscr{X}}$, whence from [LNS, p. 28, Prop. 2.5.8 (a)], $\mathbf{R}\mathscr{H}om^\bullet(\mathbf{R}\varGamma'_{\widehat{\mathscr{X}}}\kappa^*\mathscr{G}, \mathscr{E}(x)[-\Delta(x)]) \in \mathbf{D}_c(\widehat{\mathscr{X}})$. This amounts to saying that the $\widehat{A}$-module $\prod_{i\in I}\widehat{A}$ is finitely generated, forcing I to be finite. We are now in a position to prove:

Proposition 6.2.5. *Let $\mathscr{X} \in \mathbb{F}$ be of finite Krull dimension and Δ a codimension function on $\mathscr{X}$.*

(a) *If $\mathscr{G}_1$, $\mathscr{G}_2$ are c-Gorenstein on $(\mathscr{X}, \Delta)$ then $\mathbf{R}\mathscr{H}om^\bullet(\mathscr{G}_1, \mathscr{G}_2)$ is $\mathbf{D}(\mathscr{X})$-isomorphic to a locally free $\mathscr{O}_{\mathscr{X}}$-module of finite rank.*

(b) *If $\mathscr{F}_1, \mathscr{F}_2 \in \mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ are t-Gorenstein on $(\mathscr{X}, \Delta)$ then $\mathbf{R}\mathscr{H}om^\bullet(\mathscr{F}_1, \mathscr{F}_2)$ is $\mathbf{D}(\mathscr{X})$-isomorphic to a locally free $\mathscr{O}_{\mathscr{X}}$-module of finite rank.*

(c) *Let $\mathscr{F} \in \mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ be t-Gorenstein, and for each $x \in \mathscr{X}$, let $r(x)$ be the number of copies in $\mathrm{H}^{\Delta(x)}_x(\mathscr{F})$ of the injective hull of the residue field $k(x)$ (regarded as a $\mathscr{O}_{\mathscr{X},x}$-module). Then $r(x)$ is constant on connected components of $\mathscr{X}$.*

Proof. Using the functors $\mathbf{\Lambda}_{\mathscr{X}}$ and $\varGamma'_{\mathscr{X}}$, especially (6.1.3) and Remark 6.1.5, we see that statements (a) and (b) are equivalent to each other. We will therefore only prove (a). Let $x \in \mathscr{X}$ be an arbitrary closed point, and as before, let $\kappa\colon \widehat{\mathscr{X}} \to \mathscr{X}$

be the corresponding completion map. We will show below in Lemma 6.2.9 that since $\mathscr{X}$ is of finite Krull dimension (and since $\mathscr{G}_1$ and $\mathscr{G}_2$ are c-Gorenstein), we have a natural isomorphism

$$\kappa^*\mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{G}_1,\, \mathscr{G}_2) \xrightarrow{\;\sim\;} \mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\widehat{\mathscr{X}}}}(\kappa^*\mathscr{G}_1,\, \kappa^*\mathscr{G}_2).$$

From (6.2.4) we get isomorphisms $\mathbf{R}\varGamma'_{\widehat{\mathscr{X}}}\kappa^*\mathscr{G}_1 \cong \oplus_{i\in I}\mathscr{E}(x)[-\Delta(x)]$ and $\mathbf{R}\varGamma'_{\widehat{\mathscr{X}}}\kappa^*\mathscr{G}_2 \cong \oplus_{j\in J}\mathscr{E}(x)[-\Delta(x)]$, where the index sets I and J are finite. We thus have a series of isomorphisms:

$$\begin{aligned}\kappa^*\mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{G}_1,\, \mathscr{G}_2) &\xrightarrow{\;\sim\;} \mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\widehat{\mathscr{X}}}}(\kappa^*\mathscr{G}_1,\, \kappa^*\mathscr{G}_2)\\ &\xrightarrow[(6.1.3)]{\;\sim\;} \mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\widehat{\mathscr{X}}}}(\mathbf{R}\varGamma'_{\widehat{\mathscr{X}}}\kappa^*\mathscr{G}_1,\, \mathbf{R}\varGamma'_{\widehat{\mathscr{X}}}\kappa^*\mathscr{G}_2)\\ &\;\;\cong\;\; \mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\widehat{\mathscr{X}}}}(\oplus_{i\in I}\mathscr{E}(x),\, \oplus_{j\in J}\mathscr{E}(x))\\ &\;\;\cong\;\; \mathscr{O}_{\widehat{\mathscr{X}}}^{|I|\cdot|J|}\end{aligned}$$

Now, by Lemma 6.2.9 (a), $\mathscr{G}_1$ and $\mathscr{G}_2$ are in $\mathbf{D}^b_{\mathrm{c}}(\mathscr{X})$, whence $\mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{G}_1,\, \mathscr{G}_2)$ has coherent cohomology. Part (a) follows since $x \in \mathscr{X}$ was an arbitrary closed point. Part (c) is an immediate consequence of the proof just given. Indeed $r(x)^2$ is the rank of the locally free sheaf $H^0(\mathscr{H}om^{\bullet}(\mathscr{F},\, \mathscr{F}))$ on the connected component containing x. □

As a corollary we get the following theorem which contains [Db, p. 125, Thm. 3.3]. (See Remark 6.2.7.)

Theorem 6.2.6. *Let $(\mathscr{X},\, \Delta) \in \mathbb{F}^r_{\mathrm{c}}$, i.e. $(\mathscr{X}, \Delta) \in \mathbb{F}$ and $\mathscr{X}$ possesses a c-dualizing complex (equivalently $\mathscr{X}$ possesses a t-dualizing complex which is in $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$). Then*

(a) *$\mathscr{G}$ is c-Gorenstein on $(\mathscr{X},\, \Delta)$ if and only if there is a $\mathbf{D}(\mathscr{X})$-isomorphism*

$$\mathscr{G} \cong \mathscr{D} \otimes \mathcal{V}$$

where $\mathscr{D}$ is a c-dualizing complex on $\mathscr{X}$ with associated codimension function Δ and $\mathcal{V}$ is a locally free $\mathscr{O}_{\mathscr{X}}$-module of finite rank.

(b) *$\mathscr{F} \in \mathbf{D}^*_{\mathrm{c}}$ is t-Gorenstein on $\mathscr{X}$ if and only if there is a $\mathbf{D}(\mathscr{X})$-isomorphism*

$$\mathscr{F} \cong \mathscr{R} \otimes \mathcal{V}$$

where $\mathscr{R}$ is a t-dualizing complex on $\mathscr{X}$ with associated codimension function Δ and $\mathcal{V}$ is a locally free $\mathscr{O}_{\mathscr{X}}$-module of finite rank.

Remark 6.2.7. This remark is intended towards commutative algebraists, and we adhere to the terminology there. In particular, on a local ring, a complex of modules is called dualizing if it is residual. Let us assume, for this remark, that A is a local ring possessing a dualizing complex, Δ_{F} is the associated fundamental codimension function and let M be a finitely generated A-module. In [Db, p. 125, Thm. 3.3], Dibaei proves the following (with the notion of S_2, unless otherwise stated, being the usual notion of S_2).:

Suppose A satisfies S_2 and suppose that it possesses a dualizing complex. Assume M satisfies S_2 and $(0 :_A M) = 0$ and let $E(M)$ denote the Cousin complex $E(M)$ of M with respect to the fundamental codimension function Δ_{F} on Spec(A). *Then $E(M)$ is an injective complex if and only if M is isomorphic to a direct sum of a finite number of copies of the canonical module K of the ring A.*

Let us prove this using Theorem 6.2.6. In view of 6) of Remarks 3.2.9, evidently

if A is S_2 and M is a direct sum of a finite number of copies of K (without the assumption that M is S_2, as noticed by Dibaei in his proof) we must have M is also S_2 and $E(M)$ is an injective complex. Conversely (without the assumption that A is S_2, again noticed by Dibaei in *loc.cit.*), if M is S_2 and $E(M)$ is an injective complex, then by Theorem 6.2.6, we have $E(M)$ is a direct sum of a finite number of copies of the fundamental dualizing complex. Taking the zeroth cohomology we deduce that M is a direct sum of a finite number of copies of K. Note that as a consequence A must be S_2. Indeed $E(K)$, being a direct summand of $E(M)$, is such that $K = H^0(E(K))$. This means K is S_2 . It follows that $\mathrm{Hom}^{\bullet}_A(K, E(K)) \neq 0$. Thus K is $\Delta_{_F}$-S_2. In view of Theorem 3.2.8 this forces A to be S_2. As the reader must have noticed, $E(K)$ is the fundamental dualizing complex in this case.

Proof of Theorem 6.2.6. First note that since $\mathscr{X}$ carries a c-dualizing complex, it is necessarily of finite Krull dimension. Clearly (via a judicious use of $\mathbf{\Lambda}_{\mathscr{X}}$ and $\mathbf{R}\varGamma'_{\mathscr{X}}$) statement (b) is equivalent to statement (a). We prove (a). To that end, suppose $\mathscr{G}$ is c-Gorenstein with respect to Δ. Let $\mathscr{D}$ be any c-dualizing complex whose associated codimension function is Δ.[6] Now $\mathscr{D}$ is c-Gorenstein with respect to Δ, whence by Proposition 6.2.5 (a), $\mathbf{R}\mathscr{H}om^{\bullet}(\mathscr{G}, \mathscr{D})$ is $\mathbf{D}(\mathscr{X})$ isomorphic to a locally free sheaf $\mathcal{W}$ of finite rank. Let $\mathcal{V}$ be the $\mathscr{O}_{\mathscr{X}}$-module dual (=vector bundle dual in this case) of $\mathcal{W}$. We then have

$$\mathscr{G} \cong \mathbf{R}\mathscr{H}om^{\bullet}(\mathcal{W}, \mathscr{D}) \cong \mathscr{D} \otimes \mathcal{V}.$$

The converse is obvious. Indeed, if $\mathscr{G} \cong \mathscr{D} \otimes \mathcal{V}$ as in the statement, then $\mathscr{G}$ is c-Gorenstein, since a c-dualizing complex is obviously c-Gorenstein with respect to its associated codimension function. $\square$

The proof of Proposition 6.2.5 is incomplete since we need to prove Lemma 6.2.9. So as before, assume $x \in \mathscr{X}$ is a closed point and $\kappa\colon \widehat{\mathscr{X}} \to \mathscr{X}$ the completion of $\mathscr{X}$ along $\{x\}$.

The adjoint relation between κ^* and κ_* (see [Sp, p. 147, Prop. 6.7]) gives us a bifunctorial map

$$\kappa^*\mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{G}_1, \mathscr{G}_2) \longrightarrow \mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\widehat{\mathscr{X}}}}(\kappa^*\mathscr{G}_1, \kappa^*\mathscr{G}_2) \qquad (\mathscr{G}_1, \mathscr{G}_2 \in \mathbf{D}(\mathscr{X})) \tag{6.2.8}$$

induced by the composite

$$\mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{G}_1, \mathscr{G}_2) \longrightarrow \mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{G}_1, \kappa_*\kappa^*\mathscr{G}_2) \xleftarrow{\ \sim\ } \kappa_*\mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_{\widehat{\mathscr{X}}}}(\kappa^*\mathscr{G}_1, \kappa^*\mathscr{G}_2).$$

Suppose that $\mathscr{G}_1$ and $\mathscr{G}_2$ are both in $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$ and one the following holds (cf. [AJL1, p. 8, (1),(2) and (3)]):

(1) $\mathscr{G}_1 \in \mathbf{D}^-_{\mathrm{c}}(\mathscr{X})$ and $\mathscr{G}_2 \in \mathbf{D}^+_{\mathrm{c}}(\mathscr{X})$; or
(2) $\mathscr{G}_2$ has finite injective dimension (i.e., $\mathscr{G}_2$ is $\mathbf{D}(\mathscr{X})$-isomorphic to a bounded $\mathcal{A}(\mathscr{X})$-injective complex; or
(3) $\mathscr{G}_1$ has local resolutions by finite locally free $\mathscr{O}_{\mathscr{X}}$-modules.

Then using a "way-out" argument as in [AJL1, p. 8]—after replacing $\mathscr{X}$ by an affine neighborhood $\mathrm{Spf}(A, I)$ of our closed point if necessary—one sees that (6.2.8) is an isomorphism (one can reduce to the case where $\mathscr{G}_1$ is bounded-above complex of locally free $\mathscr{O}_{\mathscr{X}}$-modules and $\mathscr{G}_2$ is a single coherent $\mathscr{O}_{\mathscr{X}}$-module). In the event $\mathscr{G}_1$ and $\mathscr{G}_2$ are c-Gorenstein Lemma 6.2.9 implies that condition (1) above is satisfied.

[6]By our hypothesis, such a $\mathscr{D}$ exists, since any translation of a c-dualizing complex is again c-dualizing, and any two codimension functions are translates of each other.

Lemma 6.2.9. *Let $\mathscr{X}$ be of finite Krull dimension and Δ a codimension function on it.*

(a) *If $\mathscr{G} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$ is such that $\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}$ is Cohen-Macaulay on $(\mathscr{X}, \Delta)$ then $\mathscr{G} \in \mathbf{D}^b_{\mathrm{c}}(\mathscr{X})$.*

(b) *If $\mathscr{G}_1, \mathscr{G}_2 \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$ are such that $\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}_i$ is Δ Cohen-Macaulay for $i = 1, 2$, then* (6.2.8) *is an isomorphism for every closed point $x \in \mathscr{X}$.*

Proof. Suppose $\mathscr{F} := \mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}$ is Δ-Cohen-Macaulay. Then $\mathscr{F}$ is bounded since it is Δ-Cohen-Macaulay and $\mathscr{X}$ has finite Krull dimension. Since $\mathscr{X}$ is noetherian, it is quasi-compact. By [AJL1, p.30, Lemma (4.3)] it follows that $\mathbf{\Lambda}_{\mathscr{X}}\mathscr{F}$ is bounded. But $\mathscr{G} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$, whence $\mathbf{\Lambda}_{\mathscr{X}}\mathscr{F} \cong \mathscr{G}$. This proves (a). Part (b) follows from the argument given before the statement of the Lemma. □

6.3. **Azumaya algebras.** We are interested in understanding the algebra of endomorphisms of a t-Gorenstein complex (in $\mathbf{D}^*_{\mathrm{c}}$) on a formal schemes. We begin a result that is well known over ordinary schemes (cf. [G, p. 57, Thm. 5.1] and [C, p. 163, Prop. 6.11.1]) and lends itself to an easy generalization over formal schemes. Recall from [LNS, p.124, 10.3] that a map of formal schemes $f\colon \mathscr{X} \to \mathscr{Y}$ is *étale* if it is smooth of relative dimension zero. It is *adic* if $\mathscr{I}\mathscr{O}_{\mathscr{Y}}$ is an ideal of definition of $\mathscr{X}$ for some (and hence every) ideal of definition $\mathscr{I} \subset \mathscr{O}_{\mathscr{Y}}$ of $\mathscr{Y}$.

Theorem 6.3.1. *Let $\mathscr{X}$ be a formal scheme, $\mathscr{A}$ a coherent $\mathscr{O}_{\mathscr{X}}$-algebra. The following conditions are equivalent:*

(i) *$\mathscr{A}$ is a locally free as a $\mathscr{O}_{\mathscr{X}}$-module, and, for every closed point $x \in \mathscr{X}$, $\mathscr{A}(x) := \mathscr{A} \otimes k(x)$ is a central simple algebra, i.e., $\mathscr{A}(x) \otimes_{k(x)} \bar{k}(x)$ is isomorphic to a matrix algebra over the algebraic closure $\bar{k}(x)$ of $k(x)$.*

(ii) *$\mathscr{A}$ is a locally free $\mathscr{O}_{\mathscr{X}}$-module, and the natural homomorphism*
$$\mathscr{A} \otimes_{\mathscr{O}_{\mathscr{X}}} \mathscr{A}^{\circ} \to \mathscr{E}nd_{\mathscr{O}_{\mathscr{X}}}(\mathscr{A})$$
is an isomorphism, where $\mathscr{A}^{\circ}$ is the opposite algebra of $\mathscr{A}$.

(iii) *For every closed point $x \in \mathscr{X}$, there exists an integer $r \geq 1$, an open neighborhood U of x, and a* finite étale surjective *map $p\colon U' \to U$, such that $\mathscr{A}_{U'} := p^*\mathscr{A}$ is isomorphic to the $r \times r$ matrix algebra $M_r(\mathscr{O}_{U'})$ over $\mathscr{O}_{U'}$.*

(iv) *The same as (iii), with $p\colon U' \to U$ merely surjective and étale adic.*

Proof. A couple of comments are in order. In (iii), the requirement that $U' \to U$ is finite forces the map to be adic. We have restricted ourselves to closed points in (i), (iii) and (iv) whereas [G, thm. 5.1] has no such restriction, but very obviously for ordinary schemes, our formulation agrees with the classical formulation(s). The statements are local and therefore we may assume $\mathscr{X} = \mathrm{Spf}(R, I)$ where (R, I) is an adic noetherian ring. Let $X = \mathrm{Spec}(R)$ and let $\kappa\colon \mathscr{X} \to X$ be the completion map. Suppose $A = \Gamma(\mathscr{X}, \mathscr{A})$. Then A is a finitely generated R-algebra and and defines an $\mathscr{O}_X$-algebra $\widetilde{A}$ (which is the same as the quasi-coherator of $\kappa_*\mathscr{A}$ (cf. [I, p. 187, Lemme 3.2] and [AJL2, p. 31, § 3.1])). Since (i)–(iv) are equivalent conditions for $\widetilde{A}$ (i.e., with $(X, \widetilde{A})$ replacing $(\mathscr{X}, \mathscr{A})$), one can deduce the same for $\mathscr{A}$. We leave the details to the reader. We point to [S, §§ 2.1, pp. 144-145] for the relationship between the local rings of X and $\mathscr{X}$, and we point out that since (A, I) is adic (i.e., A is complete with respect I), I is in the Jacobson radical of A. □

Remark 6.3.2. See [G, § 2, p. 51 and Remarque 5.12, p. 60] for generalizations to locally ringed toposes.

Definition 6.3.3. Let $\mathscr{X}$ be a formal scheme and $\mathscr{A}$ a sheaf of $\mathscr{O}_{\mathscr{X}}$-algebras on $\mathscr{X}$. We say $\mathscr{A}$ is an *Azumaya algebra* on $\mathscr{X}$ if it satisfies any of the equivalent conditions of Theorem 6.3.1. An Azumaya algebra $\mathscr{A}$ is said to be *split* if it is isomorphic (as an $\mathscr{O}_{\mathscr{X}}$-algebra) to the algebra of endomorphisms $\mathscr{E}nd_{\mathscr{O}_{\mathscr{X}}}(\mathcal{V})$ of a locally free $\mathscr{O}_{\mathscr{X}}$-module of finite rank $\mathcal{V}$.

Let $(\mathscr{X}, \Delta) \in \mathbb{F}_c$ with $\mathscr{X}$ of finite Krull dimension. For a t-Gorenstein complex $\mathscr{F} \in \mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ on $(\mathscr{X}, \Delta)$, we have a sheaf of $\mathscr{O}_{\mathscr{X}}$-algebras $\mathscr{A} = \mathscr{A}(\mathscr{F})$ given by

$$\mathscr{A}(\mathscr{F}) := \mathscr{H}om_{\mathbf{C}(\mathscr{X})}(E_\Delta(\mathscr{F}), E_\Delta(\mathscr{F})). \tag{6.3.4}$$

Remarks 6.3.5. In what follows $\mathscr{F} \in \mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ is t-Gorenstein and $\mathscr{G} = \boldsymbol{\Lambda}_{\mathscr{X}}\mathscr{F}$.

1) Very clearly, if $\mathscr{F} \in \mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ is t-Gorenstein, then

$$\mathscr{A}(\mathscr{F}) = \mathscr{A}(E_\Delta(\mathscr{F})).$$

We could therefore have restricted ourselves to to complexes in $\mathbf{coz}_\Delta(\mathscr{X})$ which are t-Gorenstein. However, allowing ourselves all t-Gorenstein complexes in $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ gives us greater flexibility. In practice (i.e., in our proofs) we will almost always identify t-Gorenstein complexes $\mathscr{F}$ with $E_\Delta(\mathscr{F})$ via the Suominen isomorphism $\mathsf{S}\colon \mathscr{F} \xrightarrow{\sim} E_\Delta(\mathscr{F})$ [LNS, p.42, Corollary 3.3.2].

2) In $\mathbf{D}(\mathscr{X})$, $\mathscr{A}(\mathscr{F})$ can be identified with $\mathbf{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{F})$. In fact we have a sequence of isomorphisms:

$$\begin{aligned}\mathscr{A}(\mathscr{F}) &:= \mathscr{H}om_{\mathbf{C}(\mathscr{X})}(E_\Delta(\mathscr{F}), E_\Delta(\mathscr{F}))\\ &\xrightarrow{\sim} H^0(\mathbf{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{F}))\\ &\xrightarrow{\sim} \mathbf{R}\mathscr{H}om^\bullet(\mathscr{F}, \mathscr{F})\end{aligned}$$

The first isomorphism is deduced by noting that any $\mathbf{C}(\mathscr{X})$-endomorphism of $E_\Delta(\mathscr{F})$ is equivalent to a $\mathbf{D}(\mathscr{X})$-endomorphism of $\mathscr{F}$ using Suominen's equivalence of categories between Cohen-Macualay complexes and Cousin complexes (see [LNS, p. 42, 3.3.1 and 3.3.2]), i.e., we have $\mathrm{Hom}_{\mathbf{C}(\mathscr{X})}(E_\Delta(\mathscr{F}), E_\Delta(\mathscr{F}))$ is isomorphic to $\mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(\mathscr{F}, \mathscr{F}) = H^0(\mathbf{R}\mathrm{Hom}^\bullet(\mathscr{F}, \mathscr{F}))$ which upon sheafifying gives us the required isomorphism. The second isomorphism follows from Proposition 6.2.5(b).

3) $\mathscr{A}(\mathscr{F})$ is therefore a locally free $\mathscr{O}_{\mathscr{X}}$-module of finite rank by Prop. 6.2.5(b).

4) By Remark 6.1.5 we have a $\mathbf{D}(\mathscr{X})$ isomorphism

$$\mathscr{A}(\mathscr{F}) \cong \mathbf{R}\mathscr{H}om^\bullet(\mathscr{G}, \mathscr{G}).$$

5) Let $x \in \mathscr{X}$ be a closed point, $\kappa = \kappa_x\colon \widehat{\mathscr{X}} \to \mathscr{X}$ the completion of $\mathscr{X}$ along $\{x\}$, $\mathfrak{m}_x$ the ideal sheaf of x, $A = \Gamma(\widehat{\mathscr{X}}, \mathscr{O}_{\widehat{\mathscr{X}}}) = \widehat{\mathscr{O}}_{\mathscr{X},x}$, $\mathfrak{m} = \mathfrak{m}_A$ the maximal ideal of A (note $\widehat{\mathscr{X}} = \mathrm{Spf}(A, \mathfrak{m})$), and Δ' the codimension function on $\widehat{\mathscr{X}}$ which gives the unique point of $\widehat{\mathscr{X}}$ the value $\Delta(x)$. Set

$$\widehat{\mathscr{F}} = \kappa^*\mathbf{R}\varGamma_{\mathfrak{m}_x}\mathscr{F}.$$

As in 1) above, we identify $\mathscr{F}$ with $E_\Delta(\mathscr{F})$. Then $\widehat{\mathscr{F}}$ is the sheafification of the complex of A-modules $\Gamma_{\mathfrak{m}_x}\mathscr{F}$ $(= \mathrm{H}^{\Delta(x)}_{\mathfrak{m}_x}(\mathscr{F})[-\Delta(x)])$.[7] This shows that $\widehat{\mathscr{F}}$ is t-Gorenstein with respect to Δ' and it is clearly in $\mathbf{D}^*_{\mathrm{c}}(\widehat{\mathscr{X}})$ since it can be identified

[7] Note that there is a difference between $\varGamma_{\mathfrak{m}_x}$ and $\Gamma_{\mathfrak{m}_x}$. More precisely, $\Gamma_{\mathfrak{m}_x}\mathscr{F} = \Gamma(\mathscr{X}, \varGamma_{\mathfrak{m}_x}\mathscr{F})$

in $\mathbf{D}(\widehat{\mathscr{X}})$ with $\mathbf{R}\varGamma'_{\widehat{\mathscr{X}}}\kappa^*\mathscr{G}$. We are now in a position to define a natural map of $\mathscr{O}_{\widehat{\mathscr{X}}}$-algebras

$$\kappa^*\mathscr{A}(\mathscr{F}) \to \mathscr{A}(\widehat{\mathscr{F}}). \tag{6.3.6}$$

Indeed, with $\mathscr{F} = E_\Delta(\mathscr{F})$, any map $\mathscr{F} \to \mathscr{F}$ in $\mathbf{C}(\mathscr{X})$ gives rise to a map of complexes $\varGamma'_{\mathfrak{m}_x}\mathscr{F} \to \varGamma'_{\mathfrak{m}_x}\mathscr{F}$ in $\mathbf{C}(\widehat{\mathscr{X}})$, and this correspondence is compatible with Zariski localizations. In other words, we have a map of $\mathscr{O}_{\mathscr{X}}$-algebras $\mathscr{A}(\mathscr{F}) \to \kappa_*\mathscr{A}(\widehat{\mathscr{F}})$, which defines (6.3.6) by the adjointness of the pair $(\kappa^*,\, \kappa_*)$.

Lemma 6.3.7. *With the notations and hypotheses of Remarks* 6.3.5*, the map* $\kappa^*\mathscr{A}(\mathscr{F}) \to \mathscr{A}(\widehat{\mathscr{F}})$ *of* (6.3.6) *is an isomorphism of* $\mathscr{O}_{\widehat{\mathscr{X}}}$*-algebras.*

Proof. Let $\mathscr{G} = \mathbf{\Lambda}_{\mathscr{X}}\mathscr{F}$. It is enough to show that (6.3.6) is an isomorphism in $\mathbf{D}(\widehat{\mathscr{X}})$. To that end note that $\mathbf{\Lambda}_{\widehat{\mathscr{X}}}\widehat{\mathscr{F}} = \kappa^*\mathscr{G}$. By Lemma 6.2.9, the natural map (6.2.8) is an isomorphism for $\mathscr{G}_1 = \mathscr{G}_2 = \mathscr{G}$, i.e. we have a natural $\mathbf{D}(\mathscr{X})$ isomorphism

$$\kappa^*\mathbf{R}\mathscr{H}om^\bullet(\mathscr{G}, \mathscr{G}) \xrightarrow{\ \sim\ } \mathbf{R}\mathscr{H}om^\bullet(\kappa^*\mathscr{G},\, \kappa^*\mathscr{G}).$$

The assertion follows from 2) and 4) of Remarks 6.3.5 applied to $\mathscr{F}, \mathscr{G}, \widehat{\mathscr{F}}$ and $\kappa^*\mathscr{G}$ $(=\mathbf{\Lambda}_{\widehat{\mathscr{X}}}\widehat{\mathscr{F}})$. □

Proposition 6.3.8. *Let* $(\mathscr{X}, \Delta) \in \mathbb{F}_c$ *with* $\mathscr{X}$ *of finite Krull dimension, and suppose* $\mathscr{F} \in \mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ *is t-Gorenstein. Then the sheaf of* $\mathscr{O}_{\mathscr{X}}$*-algebras* $\mathscr{A}(\mathscr{F})$ *of* (6.3.4) *is an* $\mathscr{O}_{\mathscr{X}}$*-Azumaya algebra.*

Proof. For a closed point $x \in \mathscr{X}$ we have, thanks to Lemma 6.3.7, a $k(x)$ algebra isomorphism

$$\mathscr{A} \otimes \otimes k(x) \xrightarrow{\ \sim\ } \mathscr{A}(\widehat{\mathscr{F}}) \otimes k(x),$$

the tensor products being over appropriate structure sheaves. By Theorem 6.3.1, we are therefore reduced to the case $\mathscr{X} = \mathrm{Spf}(A, \mathfrak{m})$, $(A, \mathfrak{m})$ being a complete local ring, and $\mathscr{F} = \oplus_{i=1}^r \mathscr{E}(x)[-\Delta(x)]$, a translate of a direct sum of a finite number of copies of the $\mathscr{O}_{\mathscr{X}}$-injective hull of $k(x)$. Since $\mathscr{E}nd_{\mathscr{O}_{\mathscr{X}}}(\mathscr{E}(x))$ is canonically isomorphic to $\mathscr{O}_{\mathscr{X}}$, it follows that $\mathscr{A}(\mathscr{F})$ is isomorphic, as an $\mathscr{O}_{\mathscr{X}}$-algebra, to the matrix algebra $M_r(\mathscr{O}_{\mathscr{X}})$. It follows that $\mathscr{A}(\mathscr{F}) \otimes k(x)$ is a matrix algebra over $k(x)$. The assertion follows from the definition of Azumaya algebras in 6.3.3 (see also Theorem 6.3.1 (i)). □

One might guess (based on results in [FFGR]) that $\mathscr{X}$ has a c-dualizing complex if (and clearly only if) it has a t-Gorenstein complex $\mathscr{F} \in \mathbf{D}^*_{\mathrm{c}}$ such that $\mathscr{A}(\mathscr{F})$ is split. This may be overly optimistic. What can be shown is that in the event $\mathscr{A}(\mathscr{F})$ is split, $\mathscr{X}$ can be covered by Zariski open subschemes each of which admits a c-dualizing complex. However, if $\mathscr{A}$ is split in a strong sense, i.e., when $\mathscr{A}$ is isomorphic to a matrix algebra $M_r(\mathscr{O}_{\mathscr{X}})$, then $\mathscr{X}$ does have a c-dualizing complex.

Theorem 6.3.9. *Let* $(\mathscr{X},\, \Delta) \in \mathbb{F}_c$ *with* $\mathscr{X}$ *of finite Krull dimension.*

(a) *Suppose* $\mathscr{X}$ *admits a c-Gorenstein complex* $\mathscr{G}$ *with respect* Δ *such that* $\mathscr{A}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G})$ *is isomorphic as an* $\mathscr{O}_{\mathscr{X}}$*-algebra to* $M_r(\mathscr{O}_{\mathscr{X}})$ *for some positive integer* r*. Then* $\mathscr{X}$ *admits a c-dualizing complex.*
(b) *Suppose* $\mathscr{X}$ *admits a t-Gorenstein complex* $\mathscr{F} \in \mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$ *with respect to* Δ *such that* $\mathscr{A}(\mathscr{F})$ *is isomorphic as an* $\mathscr{O}_{\mathscr{X}}$*-algebra to* $M_r(\mathscr{O}_{\mathscr{X}})$ *for some positive integer* r*. Then* $\mathscr{X}$ *admits a c-dualizing complex.*

Proof. Clearly (a) and (b) are equivalent. We will prove (b). To that end, we make, without loss of generality, the simplifying assumption that $\mathscr{F} = E_\Delta(\mathscr{F})$. Now $M_r(\mathscr{O}_\mathscr{X})$ has idempotent sections $N_1, \dots, N_r$ such that $N_1 + \cdots + N_r = 1$, $N_i^2 = N_i$, $i = 1, \dots, r$, and $N_iN_j = 0$ for $i \neq j$ and $1 \leq i, j \leq r$. These give r idempotent endomorphisms in $\mathbf{C}(\mathscr{X})$

$$\varphi_i : \mathscr{F} \to \mathscr{F} \qquad (i = 1, \dots r)$$

such that $\varphi_1 + \cdots + \varphi_r = 1 \in \mathscr{A}(\mathscr{F})$, $\varphi_i^2 = \varphi_i$ for $i = 1, \dots, r$, and $\varphi_i\varphi_j = 0$ for $i \neq j$ and $1 \leq i, j \leq r$. These splitting idempotents break up $\mathscr{F}$ into direct summands $\mathscr{F}_i$, $i = 1, \dots, r$. In fact $\mathscr{F}_i$ is the image of $\mathscr{F}$ under φ_i as well as the kernel of the sum of the φ_j distinct from φ_i. Direct summands of injectives being injective, and subcomplexes of Cousin complexes being Cousin, one sees that the $\mathscr{F}_i$ are t-Gorenstein and for and $x \in \mathscr{X}$, $\mathscr{F}(x)$ consists of only one copy of the injective hull of $k(x)$. We claim that $\mathscr{F}_i \in \mathbf{D}_\mathrm{c}^*$ for every i. But $\mathbf{\Lambda}_\mathscr{X}(\mathscr{F}_i)$ can be identified with a direct summand of $\mathbf{\Lambda}_\mathscr{X}(\mathscr{F})$, whence the former has coherent cohomology (since the latter has). This proves that $\mathscr{F}_i \in \mathbf{D}_\mathrm{c}^*$ for $i = 1, \dots, r$.

Using the matrices M_{ij} which have 1 in the (i, j)-th spot and zero elsewhere, we get maps $\varphi_{ij} : \mathscr{F} \to \mathscr{F}$. Restricting φ_{ij} to $\mathscr{F}_j$ and projecting to $\mathscr{F}_i$, we get maps $\psi_{ij} : \mathscr{F}_j \to \mathscr{F}_i$. It is easy to check that these are isomorphisms, and that the sheaf of endomorphisms, $\mathscr{H}om_{\mathbf{C}(\mathscr{X})}(\mathscr{F}_i, \mathscr{F}_j)$ is isomorphic to $\mathscr{O}_\mathscr{X}$ for every $1 \leq i, j \leq r$. This proves that the $\mathscr{F}_i$ are residual. Since they are in $\mathbf{D}_\mathrm{c}^*(\mathscr{X})$, it follows that the $\mathbf{\Lambda}_\mathscr{X}\mathscr{F}_i$ are c-dualizing on $\mathscr{X}$. □

Theorem 6.3.1(iii), Theorem 6.3.8 and Theorem 6.3.9 gives immediately part (a) of the following theorem generalizing [FFGR, p. 209, Cor. (4.8)]. Part (b) follows from [LNS, p. 109, Prop. 9.3.5] and part (a), using the fact that $f^*E_\Delta \cong E_{\Delta'}f^*$ for an étale adic map $f : (\mathscr{X}', \Delta') \to (\mathscr{X}, \Delta)$.

Theorem 6.3.10. *Suppose $(\mathscr{X}, \Delta)$ has a c-Gorenstein complex which non-exact on every connected component of $\mathscr{X}$ (or equivalently a t-Gorenstein complex in $\mathbf{D}_\mathrm{c}^*(\mathscr{X})$, which is non-exact on every connected component of $\mathscr{X}$) and suppose $\mathscr{X}$ is of finite Krull dimension.*

(a) *For every closed point $x \in \mathscr{X}$ we can find a Zariski open neighborhood U of x and a finite étale map $U' \to U$ such that U' has a c-dualizing complex.*
(b) *We have $E_\Delta\mathbf{D}_\mathrm{c}^*(\mathscr{X}) \subset \mathbf{D}_\mathrm{c}^*(\mathscr{X})$. In particular, if $\mathscr{X}$ is an ordinary scheme, and $\mathscr{F}$ has coherent cohomology, then $E_\Delta\mathscr{F}$ has coherent cohomology.*

Remarks 6.3.11. 1) Examining the proof of Theorem 6.3.9 we notice that for a closed point $x \in \mathscr{X}$, $\mathscr{A}(\mathscr{F}) \otimes k(x)$ is already a split Azumaya algebra over $k(x)$ (i.e., a split central simple algebra over $k(x)$). This means that the finite étale map $U' \to U$ (U a Zariski neighborhood of x) can be chosen so that $k(x') = k(x)$ for every point $x' \in U'$ lying over x. (See [G, pp. 58–59, 5.4–5.8].)

2) Our main results concerning c-Gorenstein complexes (especially Theorem 6.3.9 and Theorem 6.3.10) rely crucially on the technical result that when $\mathscr{X}$ is of finite Krull dimension (6.2.8) is an isomorphism for every closed point of $\mathscr{X}$ (see Lemma 6.2.9). On the other hand, with the benefit of hindsight, if $\mathscr{X}$ is a scheme admitting a c-Gorenstein complex, and is such that (6.2.8) is an isomorphism for every closed point $x \in \mathscr{X}$, then the conclusion of Theorem 6.3.10 holds. But any scheme containing a c-dualizing complex is of finite Krull dimension, and a little thought shows that this forces the quasi-compact scheme $\mathscr{X}$ to have finite Krull dimension.

Theorem 6.3.10 gives us the following (cf. again [FFGR, Cor. (4.8)]):

Corollary 6.3.12. *Suppose A is a Hensel local ring, complete with respect to an ideal $I \subset A$, and let $\mathscr{X} = \mathrm{Spf}(A, I)$. The following are equivalent:*

(a) *$\mathscr{X}$ has a codimension function Δ and $(\mathscr{X}, \Delta)$ has a c-Gorenstein complex.*
(b) *$\mathscr{X}$ has a c-dualizing complex.*
(c) *$\mathscr{X}$ has a codimension function Δ and $(\mathscr{X}, \Delta)$ has a t-Gorenstein complex in $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$.*
(d) *$\mathscr{X}$ has a t-dualizing complex in $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$.*

Proof. Observe that if $\mathscr{X}$ has a c-dualizing complex then it necessarily has a codimension function. For the rest, first note that (a) and (c) are equivalent, as are (b) and (d). Now let $k = A/\mathfrak{m}_A$ be the residue field of A. Now since A is Hensel any étale neighborhood of $\mathrm{Spec}(A)$ having a k-valued point over the closed point of $\mathrm{Spec}(A)$ admits a section through that point (whence so does any adic étale neighborhood of $\mathrm{Spf}(A, I)$ having a k-valued point). The equivalence of (a) and (b) is now immediate, in view of the Theorem 6.3.10 and 1) of Remarks 6.3.11. □

Remarks 6.3.13. We wish to re-interpret our results in commutative algebraic terms. If (A, I) is an adic ring (always noetherian and of finite Krull dimension in these remarks) then we will move from complexes of A-modules and sheaves on $\mathrm{Spf}(A, I)$ in the usual fashion. The terminology used in this remark is self-explanatory. For example, and if we say that a complex $M^\bullet$ of A-modules lies in $\mathbf{D}^*_{\mathrm{c}}(A, I)$, then what is meant is that the corresponding complex on $\mathrm{Spf}(A, I)$ is in $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$. In the event an ideal of definition I in A is not specified, we take $I = 0$, so that $\mathrm{Spf}(A, I) = \mathrm{Spec} A$. In these remarks, all rings and algebras occurring are commutative.

1) Suppose (A, I) is an adic ring, Δ a codimension function on (A, I), $M^\bullet$ a complex whose cohomologies are finite A-modules, and $M^\bullet \to J^\bullet$ an injective resolution of $M^\bullet$. If the complex $\Gamma_I J^\bullet$ is quasi-isomorphic to its Cousin complex $E^\bullet = E_\Delta(\mathbf{R}\Gamma_I M^\bullet)$ with respect to Δ and $E^\bullet$ consists of A-injectives, then for every maximal ideal $\mathfrak{m}$ of A, there is an element $f \in A \setminus \mathfrak{m}$ and a *finite étale* A_f-algebra B such that B possesses a dualizing complex $D^\bullet_B$ (in the usual sense). Moreover, in $\mathbf{D}(B)$,we have

$$M^\bullet \otimes_A B \cong D^\bullet_B \otimes P$$

where P is a finitely generated *projective* B-module.

2) With the above hypotheses, since B possesses a dualizing complex, $\mathrm{Spec} B$ has a codimension function, and it is not hard to see that whence so has $\mathrm{Spec} A$. We add that $D^\bullet_B$ is a c-dualizing complex on (B, IB) and $\mathbf{R}\Gamma_I D^\bullet_B$ is t-dualizing on (B, IB) and is in $\mathbf{D}^*_{\mathrm{c}}(B, IB)$.

3) Suppose A is a ring and $I \subset A$, with A not necessarily complete with respect to I. Suppose that A/I admits a codimension function Δ and that we have a complex $M^\bullet$ with finitely generated cohomology modules satisfying the hypotheses in 1) with respect to I and Δ (note that $\Gamma_I J^\bullet$ being supported on $V(A/I)$, we can talk about its Cousin with respect to Δ). Let $\widehat{A}$ be the completion of A with respect to I. Then, for every maximal ideal $\mathfrak{m}$ of $\widehat{A}$, there is an element $f \in A \setminus \mathfrak{m}$ and a finite étale A_f-algebra B such that B possesses a dualizing complex $D^\bullet_B$ and in $\mathbf{D}(B)$ we have

$$M^\bullet \otimes_A B \cong D^\bullet_B \otimes_B P$$

where P is a finitely generated projective B-module.

4) The case of $I = \mathfrak{m}_A (= \mathfrak{m})$, a maximal ideal of A, is then quite interesting. With $M^\bullet$, $J^\bullet$ as above, the previous statement reduces to the following: if $\mathrm{H}^i(\Gamma_{\mathfrak{m}} J^\bullet) = 0$ for $i \neq \Delta(\mathfrak{m})$ and $\mathrm{H}^{\Delta(\mathfrak{m})}(\Gamma_{\mathfrak{m}} J^\bullet)$ is an injective A-module, then $M^\bullet \otimes_A \widehat{A}$ is isomorphic in $\mathbf{D}(\widehat{A})$ to a direct sum of a finite number of copies of a dualizing complex $D^\bullet$ on $\widehat{A}$.

6.4. **Gorenstein modules.** Let A be a noetherian ring of finite Krull dimension. Recall, from [Sh1, p. 123], that a non-zero finite module M is called Gorenstein if its Cousin complex $E(M)$, with respect to the height filtration of M, is an injective resolution (necessarily minimal) of M, i.e., $E(M)$ consists of injectives, and $H^0(E(M)) = M$. In such a case, according to a result of Foxby [F], $\mathrm{Hom}_A(M, M)$ is a projective A-module. Moreover, A is Cohen-Macaulay [Sh1, p. 126, Cor. 3.9]. Further $\mathrm{Supp}_A M = \mathrm{Spec}(A)$. We end the paper with a discussion connecting our results with these (well known) results on Gorenstein modules. The claim is not that our proofs are simpler, but we hope they are illuminating, as perhaps any set of proofs involving a different point of view will tend to be. Since we work on the ordinary scheme $\mathrm{Spec}(A)$, the notions of c-Gorenstein complexes and t-Gorenstein complexes in $\mathbf{D}_{\mathrm{c}}^*(A)$ coincide and we simply use Gorenstein (without adornments) while referring to such complexes.

Fix $\mathfrak{p} \in \mathrm{Spec}(A)$. With $i = \mathrm{ht}_M(\mathfrak{p})$, set $E(\mathfrak{p}) := \mathrm{H}^i_{\mathfrak{p}}(M_{\mathfrak{p}})$, i.e., $E(\mathfrak{p})$ is the summand of the total module of $E(M)_{\mathfrak{p}}$ which has $\mathfrak{p}$ as its associated prime.

One can show in the usual way, for example by using [B, Lemma (3.1)][8], that $\mathfrak{p} \mapsto \mathrm{ht}_M(\mathfrak{p})$ is a codimension function on $\mathrm{Supp}_A(M)$. In fact, as we shall shortly see, $\mathrm{Supp}_A(M) = \mathrm{Spec}(A)$, whence ht_M is a codimension function and $E(M)$ is a Gorenstein complex.

Arguing as we did in the proof of Proposition 6.2.5 (bearing in mind that A has finite Krull dimension) we see that for $\mathfrak{m} \in \mathrm{Supp}_A(M)$ a maximal ideal, and $\widehat{A}$ the $\mathfrak{m}$-adic completion of A, we have

$$\mathrm{Hom}^\bullet_A(E(M),\, E(M)) \otimes_A \widehat{A} \xrightarrow{\ \sim\ } \mathrm{Hom}^\bullet_{\widehat{A}}(E(\mathfrak{m}),\, E(\mathfrak{m})).$$

The right side has no cohomology in positive degrees therefore neither does the left side. Taking the zeroth cohomology we get

$$\mathrm{Hom}_A(M,\, M) \otimes_A \widehat{A} \xrightarrow{\ \sim\ } \mathrm{Hom}_{\widehat{A}}(E(\mathfrak{m}),\, E(\mathfrak{m})).$$

But if $I = 0 :_A M$, then the left side is killed by $\widehat{I} := I\widehat{A}$. The right side is a free $\widehat{A}$-module. It follows that $\widehat{I} = 0$, whence $I = 0$. In other words, $\mathrm{Supp}_A M = \mathrm{Spec}(A)$. In particular ht_M is a codimension function.

In order to show that A is Cohen-Macaulay, it is enough to show this is so under the assumption that A has a dualizing complex. Indeed, $E(M)$ is a Gorenstein complex, therefore, and therefore, according to Theorem 6.3.10, we can find $f_1, \ldots, f_n \in A$, with $f_1 + \cdots + f_n = 1$, and finite étale algebras $A_{f_i} \to B_i$, $i = 1, \ldots, n$, such that B_i possess dualizing complexes. Moreover, with $M_i = M \otimes_A B_i$, it is easy to see that $E(M_i) = E(M) \otimes_A B_i$ is an injective resolution of M_i, whence M_i is Gorenstein as a B_i-module. If B_i is Cohen-Macaulay, so is A_{f_i}, and if this is so for every $i = 1, \ldots, n$, clearly A is Cohen-Macaulay (since $f_1 + \cdots + f_n = 1$).

[8]The statement is that for any finite A-module N, any non-negative integer i, and any immediate specialization $\mathfrak{p} \mapsto \mathfrak{q}$, the Bass number $\mu^i(\mathfrak{p}, N) \neq 0$ only if the Bass number $\mu^{i+1}(\mathfrak{q}, N) \neq 0$.

We therefore assume, without loss of generality, that A possesses a dualizing complex $D^\bullet$ (as before, following commutative algebra conventions, $D^\bullet$ is residual). Further, by translating $D^\bullet$, we may assume that the associated codimension function is ht_M. By Theorem 6.2.6, there is a projective A-module P and a $\mathbf{C}(A)$-isomorphism:

$$E(M) \xrightarrow{\sim} D^\bullet \otimes_A P.$$

Since $E(M)$ resolves M and P is projective, it follows that $H^i(D^\bullet) = 0$ for $i \neq 0$ and the natural map $K = H^0(D^\bullet) \to D^\bullet$ is a resolution. It follows that A is Cohen-Macaulay.

If the reader wishes to test her/his hold on the techniques of [AJL2], especially as used above, then the following exercise is recommended:

Exercise: Let $(A, \mathfrak{m})$ be a complete local ring, and $M \neq 0$ a finite A-module such that $\mathrm{H}^j_{\mathfrak{m}}(M)$ is an injective A-module for some $j \geq 0$ and $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i \neq j$. Show that M is a Gorenstein A-module. [*Hint:* Show that $\mathbf{R}\Gamma_{\mathfrak{m}}(M)$ is t-Gorenstein on the adic ring $(A, \mathfrak{m})$ for a suitable codimension function on $\mathrm{Spf}(A, \mathfrak{m})$. Conclude, using $\mathbf{\Lambda}_{\mathscr{X}} \varGamma'_{\mathscr{X}}(\mathscr{G}) \cong \mathscr{G}$ for $\mathscr{G} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$ ($\mathscr{X}$ any formal scheme), that M is c-Gorenstein on $(A, \mathfrak{m})$. Use Theorem 6.2.6 to conclude that M is a Gorenstein module.]

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Chennai Mathematical Institute, Sipcot IT Park, Siruseri, TN-603103, INDIA
E-mail address: `snayak@cmi.ac.in`

Department of Mathematics, East Carolina University, Greenville, NC 27858, USA
E-mail address: `sastryp@ecu.edu`